CHAPTER 18

VECTOR ANALYSIS

18.1 Line Integrals

PREREQUISITES

1. Recall how to differentiate vector functions (Section 14.6).
2. Recall how to compute work for forces described in one variable (Section 9.5).
3. Recall how to describe a curve with parametric equations (Sections 2.4 and 10.4).

PREREQUISITE QUIZ

1. Compute $(d/dt)[t^3\underline{i} + (\cos t)\underline{j} - (e^t)\underline{k}]$.
2. Compute $(d/dx)[(\ln x)\underline{i} - (2x)\underline{j} + \underline{k}]$.
3. Calculate the work done by a force $F(x) = x^2$ if $0 \leqslant x \leqslant 2$.
4. Write a set of parametric equations to describe the unit circle.
5. Suppose that $y = 2x + 3$ and $y(t) = t - 3$, find x as a function of t .

GOALS

1. *Be able to set up and compute a line integral and understand its relationship to work.*

STUDY HINTS

1. Line integrals and work. Know that the line integral's physical interpretation is work. It is given by the formula $\int_{t_1}^{t_2} \underline{\Phi}(\underline{\sigma}(t))\cdot\underline{\sigma}'(t)dt$ or $\int_{t_1}^{t_2} \underline{F}\cdot\underline{v}\, dt$. Other notations that will be used include $\int_C \underline{\Phi}(\underline{r})\cdot d\underline{r}$, $\int_C \underline{\Phi}$, and $\int_C(a\, dx + b\, dy + c\, dz)$.
2. Computing work. If you're given a vector field $\underline{\Phi}$ and an oriented path $\underline{\sigma}$, computing the line integral (or work) is a simple matter of substitution. See Example 1. If a curve comes in pieces, one can integrate over each segment and add up the results from each segment. See Example 8.
3. Sign interpretation. Positive work means that the force field did a net amount of work; i.e., the motion of the particle is in the direction of the force. If work is negative, then this amount of work is done by the particle on the force field.
4. Parametrization of curves. As long as a curve is parametrized with the correct direction and traces out the curve the correct number of times, the value of a line integral is independent of which parametrization is chosen. You can substitute for the endpoints to be sure the direction is correct. Example 5(a) shows that opposite directions yield opposite signs. Be sure the path is traversed the correct number of times. Example 5(b) shows how this can be a problem.

SOLUTIONS TO EVERY OTHER ODD EXERCISE

1. Use the formula $W = \int_{t_1}^{t_2} \underline{\Phi}(\underline{\sigma}(t))\cdot\underline{\sigma}'(t)dt$. We have $\underline{\sigma}(t) = 3t^2\underline{i} + t\underline{j} + \underline{k}$, $\underline{\sigma}'(t) = 6t\underline{i} + \underline{j}$, and $\underline{\Phi}(\underline{\sigma}(t)) = 3t^2\underline{i} + t\underline{j}$. Thus, the work done is $\int_0^1 (3t^2\underline{i} + t\underline{j})\cdot(6t\underline{i} + \underline{j})dt = \int_0^1 (18t^3 + t)dt = (9t^4/2 + t^2/2)\big|_0^1 = 5$.

5. At $(1,0,0)$, the kinetic energy, $K = 0$. At $(1,0,1)$, it is $(1/2)m\underline{v}\cdot\underline{v} = 5m/2 = 5/2$, since $m = 1$. Thus, the total work done is $5/2$. The work done by the force field is $\int_0^1 \underline{\Phi}(\underline{\sigma}(t))\cdot\underline{\sigma}'(t)dt$, where $\underline{\sigma}(t) = \underline{i} + t\underline{k}$, $\underline{\sigma}'(t) = \underline{k}$, and $\underline{\Phi}(\underline{\sigma}(t)) = \underline{k}$. Thus, $\int_0^1(\underline{k}\cdot\underline{k})dt = 1$. Therefore, you did $3/2$ units of work.

9. By using the formula $W = \int_{t_1}^{t_2} \underline{\Phi}\ (\underline{\sigma}(t))\cdot\underline{\sigma}'(t)dt$, we have $\underline{\sigma}(t) = (\cos t, \sin t)$, $\underline{\sigma}'(t) = (-\sin t, \cos t)$, and $\underline{\Phi}(\underline{\sigma}(t)) = (-\sin t, \cos t)$. Thus, the work done is $\int_0^{\pi} (\sin^2 t + \cos^2 t)dt = \pi$.

13. The integral of $\underline{\Phi}$ along $\underline{\sigma}(t)$ for $t_1 \leqslant t \leqslant t_2$ is $\int_{t_1}^{t_2} \underline{\Phi}(\underline{\sigma}(t))\cdot\underline{\sigma}'(t)dt$. $\underline{\Phi}(\underline{\sigma}(t)) = \sin t\underline{i} + \cos t\underline{j} + t\underline{k}$; $\underline{\sigma}'(t) = \cos t\underline{i} - \sin t\underline{j} + \underline{k}$. Thus, the line integral is $\int_0^{2\pi} (\sin t\underline{i} + \cos t\underline{j} + t\underline{k}) \cdot (\cos t\underline{i} - \sin t\underline{j} + \underline{k})dt = \int_0^{2\pi} t\ dt = (t^2/2)\Big|_0^{2\pi} = 2\pi^2$.

17. The line integral of $\underline{\Phi}$ along $\underline{\sigma}(t)$ is $\int_{t_1}^{t_2} \underline{\Phi}(\underline{\sigma}(t))\cdot\underline{\sigma}'(t)dt$.

(a) $\underline{\Phi}(\underline{\sigma}(t)) = \sin t\underline{i} + \cos t\underline{j} + \sin^3 t\underline{k}$; $\underline{\sigma}'(t) = \cos t\underline{i} + 2t\underline{j} + \underline{k}$, so the line integral is $\int_0^{2\pi}(\sin t \cos t + 2t \cos t + \sin^3 t)dt$. Let $u = \sin t$ to get $\int_0^{2\pi} \sin t \cos t\ dt = \int_0^0 u\ du = 0$. Let $u = \cos t$ to get $\int_0^{2\pi} \sin^3 t\ dt = \int_0^{2\pi} \sin t(1 - \cos^2 t)dt = -\int_1^1(1 - u^2)du = 0$. Integrate by parts to get $\int_0^{2\pi} 2t \cos t\ dt = 2t \sin t\Big|_0^{2\pi} - \int_0^{2\pi} 2 \sin t\ dt = 0 + 2 \cos t\Big|_0^{2\pi} = 0$. Therefore, the line integral is 0.

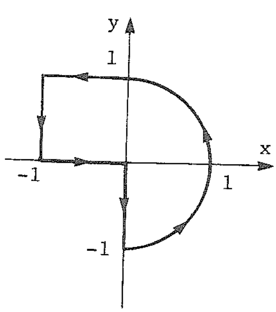
21. The parametric form of the line is $\underline{\sigma}(t) = (t,t,t)$ for $0 \leqslant t \leqslant 1$; therefore, $\underline{\Phi}(\underline{\sigma}(t)) = t^2\underline{i} + t^2\underline{j} + \underline{k}$ and $\underline{\sigma}'(t) = \underline{i} + \underline{j} + \underline{k}$. Thus, the line integral is $\int_0^1(t^2 - t^2 + 1)dt = t\Big|_0^1 = 1$.

25. In this case, we have $dx = -3 \cos^2\theta \sin\theta$, $dy = 3 \sin^2\theta \cos\theta$; and $dz = 1$, and so, substituting $x = (\cos^3\theta)d\theta$, $y = (\sin^3\theta)d\theta$, and $z = \theta d\theta$, we get $\int_C [\sin z\ dx + \cos z\ dy - (xy)^{1/3}dz] = \int_0^{7\pi/2}(-3 \cos^2\theta \sin^2\theta + 3 \sin^2\theta \cos^2\theta - \cos\theta \sin\theta)d\theta = -\int_0^{7\pi/2} \cos\theta \sin\theta\ d\theta = [(1/2)\sin^2\theta]\Big|_0^{7\pi/2} = -1/2$.

29. $f(\underline{\sigma}(t)) = \sin t + \cos t + t\cos t$; $\underline{\sigma}'(t) = \cos t\underline{i} - \sin t\underline{j} + \underline{k}$, so $\|\underline{\sigma}'(t)\| = \sqrt{\cos^2 t + (-\sin t)^2 + (1)^2} = \sqrt{2}$. Thus, the line integral is $\sqrt{2}\int_0^{2\pi} (\sin t + \cos t + t\cos t)dt = \sqrt{2}[-\cos t|_0^{2\pi} + \sin t|_0^{2\pi} + \int_0^{2\pi} t\cos t\, dt] = \sqrt{2}\int_0^{2\pi} t\cos t\, dt$. Integrate by parts with $u = t$ and $v = \sin t$: $\sqrt{2}[t\sin t|_0^{2\pi} - \int_0^{2\pi} \sin t\, dt] = \sqrt{2}\cos t|_0^{2\pi} = 0$.

33. $f(\underline{\sigma}(t)) = 6t^2$; $\underline{\sigma}'(t) = \underline{i} + 3\underline{j} + 2\underline{k}$, so $\|\underline{\sigma}'(t)\| = \sqrt{(1)^2 + (3)^2 + (2)^2} = \sqrt{14}$. Thus, the line integral is $\sqrt{14}\int_1^3 6t^2 dt = \sqrt{14}(2t^3)|_1^3 = 52\sqrt{14}$.

SECTION QUIZ

1. 

A particle traverses the "whistle" (composed of a semicircle and line segments) in a counterclockwise direction. The force acting on the particle is $\underline{F}(x,y) = 3\underline{i} + xy\underline{j}$.

(a) Compute the line integral of $\underline{F}$ along the "whistle" if it is traversed once starting at the origin.

(b) Interpret the sign and the physical meaning of your answer to part (a).

(c) Is the line integral different if the beginning and ending point is $(0,-1)$ rather than the origin? Explain.

(d) Is the line integral in part (a) different if the curve is traversed two times in a clockwise direction? Explain.

2. Compute $\int_C (x\,dx + y\sin z\,dy - e^{x/z}dz)$, where C is parametrized by $(x,y,z) = (t^2, \sin t, t)$, $0 \leqslant t \leqslant 2$.

3. It was love at first sight for Handsome Harold and Lovely Lisa. On the evening of their first meeting, Cupid instantly shot his magic arrow through their hearts and united them. Witnesses say that the arrow travelled an unusual path: $\begin{cases} (t,0) & 0 \leqslant t \leqslant 2 \\ (1 - \cos(t-2+\pi), \sin(t-2+\pi)) & 2 \leqslant t \leqslant 2+\pi \end{cases}$. Suppose that a force field of $-10\underline{j}$ opposed the arrow. Was any work required to develop this romance, i.e., how much work was done by the magical love arrow?

ANSWERS TO PREREQUISITE QUIZ

1. $3t^2\underline{i} - (\sin t)\underline{j} - e^t\underline{k}$
2. $(1/x)\underline{i} - 2\underline{j}$
3. $8/3$
4. $x = \sin t$, $y = \cos t$, $0 \leqslant t \leqslant 2\pi$
5. $x = (t - 6)/2$

ANSWERS TO SECTION QUIZ

1. (a) $1/2$

   (b) Work done by force field is $1/2$, work done by particle is $-1/2$.

   (c) No; it is just a reparametrization.

   (d) Yes; it is -1 .
2. $9 + (\sin^3 2)/3 - e^2$
3. 0

18.2 Path Independence

PREREQUISITES

1. Recall how to compute work by using line integrals (Section 18.1).
2. Recall how to compute a gradient vector (Section 16.1).
3. Recall how to compute partial derivatives (Section 15.1).

PREREQUISITE QUIZ

1. A force $xy\underline{i} + 3\underline{j}$ acts on a particle which moves from $(0,0)$ to $(1,1)$. How much work is done if the particle moves along the path:

   (a) $y = x$.

   (b) straight line segments from $(0,0)$ to $(1,0)$ and then to $(1,1)$.
2. If $f = 2 - 3xy$, what is the gradient of f?
3. If $f(x,y,z) = 3x^2yz^{1/2}$, compute $\partial f/\partial x$, $\partial f/\partial y$, and $\partial f/\partial z$.

GOALS

1. Be able to determine if a vector field is conservative or not.
2. Be able to find the antiderivative of conservative vector fields.

STUDY HINTS

1. Conservative defined. A vector field is said to be conservative if its line integral doesn't depend on the path travelled, but only on the beginning and ending positions. Since the sign of the line integral changes with a change in the direction of movement, one conclusion that can be drawn is that the line integral of a conservative field around a closed curve is zero. The converse is not true; a single zero line integral does not imply that a vector field is conservative. (But if every line integral is zero, the field is conservative.)

2. Gradients are conservative. You must know this fact. If a vector field is a gradient, then the fact that its line integral depends only on the endpoints of the integration path is clear from the formula: $\int_C \underline{\nabla} f(\underline{r})\cdot d\underline{r} = f(B) - f(A)$, where A and B are the endpoints. The converse is also true. By definition, f is called the antiderivative. Only certain vector functions, namely gradients, have antiderivatives, since there may not be a single f that satisfies both $\partial f/\partial x = a$ and $\partial f/\partial y = b$ simultaneously.

3. Cross-derivative test. If a vector field $a(x,y)\underline{i} + b(x,y)\underline{j}$ is conservative, then $a = f_x$ and $b = f_y$ for some f . By the equality of mixed partials, we get $a_y = f_{xy} = f_{yx} = b_x$. Therefore, if a vector field is conservative, then $a_y = b_x$. The cross-derivative test on p. 898 asserts the converse. Knowing the little argument given for $a_y = b_x$ is useful to help you remember this test. Example 7 shows the importance of continuity. Exercise 30 extends this test to three variables. Again, the equality of mixed partials is used.

4. Finding antiderivatives. You need to know how to find an antiderivative. Example 5 shows one method. Another method is to integrate each term and add on a constant which depends on the other variables. All of the antiderivatives should be equal. For example, suppose that the integrand is $4xyz\,dx + (2x^2z + e^y)dy + (2x^2y + 1)dz$. Integration in x yields $2x^2yz + C(y,z)$. Integrating in y gives $2x^2yz + e^y + C(x,z)$, and integrating in z gives $2x^2yz + z + C(x,y)$. Comparing terms yields $f(x,y,z) = 2x^2yz + e^y + z + C$. Note that $C(y,z) = e^y + z + C$; $C(x,z) = z + C$, and $C(x,y) = e^y + C$. Learn the method that is easiest for you.

SOLUTIONS TO EVERY OTHER ODD EXERCISE

1. The line integral of $\underline{\Phi}$ is path-independent because the vector field is conservative. The paths AOF and AODEF have the same endpoints, so the line integral is 3.

5. A nonzero line integral around a closed curve shows that the vector field is not conservative. Let C consist of three paths: C_1 is $(0,t,0)$, $0 \leqslant t \leqslant 1$; C_2 is $(t-1,1,0)$, $1 \leqslant t \leqslant 2$; C_3 is $(3-t,3-t,0)$, $2 \leqslant t \leqslant 3$. On C_1, C_2, and C_3, $\underline{\sigma}'(t)$ is $(0,1,0)$, $(1,0,0)$, and $(-1,-1,0)$, respectively. Thus, $\int_C \underline{\Phi}(\underline{\sigma}(t))\cdot\underline{\sigma}'(t)dt =$ $\int_0^1 (t,t,1)\cdot(0,1,0)dt + \int_1^2 (1,1,1)\cdot(1,0,0)dt + \int_2^3 (3-t,3-t,1)\cdot$ $(-1,-1,0)dt = \int_0^1 t\,dt + \int_1^2 dt + \int_2^3 (2t-6)dt = (t^2/2)\big|_0^1 + t\big|_1^2 +$ $(t^2 - 6t)\big|_2^3 = 1/2$. Thus, $\underline{\Phi}$ is not conservative.

9. We use the fact that if $\underline{\Phi}$ is a gradient of $f(x,y)$, then $\int_C \underline{\Phi} =$ $f(B) - f(A)$, where A and B are the endpoints of C. We recognize $2xy\,dx + x^2dy$ as the gradient of x^2y. $A = (1,0)$ and $B = (0,5\sqrt{2}/2)$, so the line integral is $f(B) - f(A) = 0$.

13. $\underline{F} = -[JMm/(x^2+y^2+z^2)^{5/2}](x\underline{i} + y\underline{j} + z\underline{k})$, so we guess this to be the gradient of $JMm(x^2+y^2+z^2)^{-3/2}$. Differentiation shows that, in fact, it is the gradient of $JMm(x^2+y^2+z^2)^{-3/2}/3 = JMmr^{-3}/3$. Since the force field is a gradient, the work or line integral, depends only on the endpoints; it is $f(B) - f(A) = JMm(r_2^{-3} - r_1^{-3})/3$.

17. We use the cross-derivative test to determine if $a(x,y)\underline{i} + b(x,y)\underline{j}$ is conservative; it is if $a_y = b_x$. If the vector is conservative, use the method of Example 5 to find an antiderivative. $a_y = 2x$ and $b_x =$ $2x$. Since $a_y = b_x$, the vector is conservative. We integrate $a(x,y) = 2xy$ with respect to x to get $x^2y + g(y)$. Differentiation with respect to y gives $x^2 + g'(y)$, and comparison with $b(x,y)$

17. (continued)

gives $g'(y) = \cos y$. Therefore, $g(y) = \sin y$, so the antiderivative is $f(x,y) = x^2y + \sin y + C$.

21. If $f(x,y) = \tan^{-1}(y/x)$, then $\partial f/\partial x = (-y/x^2)[1/(1 + (y/x)^2] = (-y/x^2)x^2/(x^2 + y^2) = -y/(x^2 + y^2)$. Also, $\partial f/\partial y = (1/x)[1/(1 + (y/x)^2)] = (1/x)x^2/(x^2 + y^2) = x/(x^2 + y^2)$. Thus, $\underline{\nabla}f$ is the vector field of Example 7.

25. We want a function f such that $\partial f/\partial x = x$, $\partial f/\partial y = y$, and $\partial f/\partial z = z$. Such a function is $(x^2 + y^2 + z^2)/2$. The work done is $f(\underline{\sigma}(b)) - f(\underline{\sigma}(a))$. $\underline{\sigma}(0) = (1,0,0)$ and $\underline{\sigma}(\pi) = (0,1,0)$; therefore, the work done is $(1/2) - (1/2) = 0$.

29. We are integrating over a gradient, so $\int_C \underline{\nabla}f(\underline{r})\cdot d\underline{r} = f(\underline{\sigma}(1)) - f(\underline{\sigma}(0)) = f(1/2,\sqrt{2}/2,3) - f(0,0,2) = (1/2)^3 - (\sqrt{2}/2)^3 + \sin(3\sqrt{2}\pi/4)$.

33. $c_y = x \exp(yz) + xyz \exp(yz) = b_z$; $c_x = y \exp(yz) = a_z$; $a_y = z \exp(yz) = b_x$. Since the three equalities are satisfied, the field must be conservative. Integrating a , b , c with respect to x , y , and z , respectively gives $x \exp(yz) + C(y,z)$, $x \exp(yz) + C(x,z)$, and $x \exp(yz) + C(x,y)$, where $C(y,z)$ is a function depending only on y and z , etc. Comparison shows that the antiderivative is $f(x,y,z) = x \exp(yz) + C$.

37. A closed curve crosses the union of circles an even number of times because, if a curve "enters" a circle, it must also "leave" the circle; therefore, the number of intersections is a multiple of two.

Pick an arbitrary fixed point P . If an arc from P to a certain region crosses the circles an even number of times, color the region red.

If the number is odd, color the region blue. Notice that a single

37. (continued)

crossing takes one from a region to an adjacent region, so adjacent regions have different colors. To see that this coloring scheme is consistent, we must verify that, if $\underline{\sigma}_1$ and $\underline{\sigma}_2$ are arcs from P to a given region, then the numbers n_1 and n_2 of crossings are both even or both odd. We must show that $n_1 - n_2$ is even, or, equivalently, that $n_1 + n_2 = n_1 - n_2 + 2n_2$ is even. But $n_1 + n_2$ is the number of crossings for a closed curve that goes along $\underline{\sigma}_1$ from P to the region and then back along $\underline{\sigma}_2$ to P. By our first observation, this number of crossings is even.

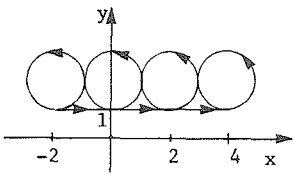
SECTION QUIZ

1. Let $f(x,y) = x^3 + \cos y$. What is $\int_C \underline{\nabla} f(\underline{r})\cdot d\underline{r}$, where C is the path along the unit circle, which is traversed $2\frac{1}{2}$ times from $(0,-1)$ to $(0,1)$? If this can't be done, explain what is missing.
2. Are the following vector fields conservative? If yes, find an antiderivative:
   (a) $(x^2y + \cos y)\underline{i} + (x^3/3 - x\sin y)\underline{j}$
   (b) $(3 + e^x)\underline{i} - y\underline{j}$
   (c) $[\exp(xy) + y\cos x]\underline{i} + [\exp(xy) + x\cos y]\underline{j}$
   (d) $(1/y)\underline{i} - (x/y^2)\underline{j}$
3. True or false: If $\int_C \underline{\Phi}\cdot d\underline{r} = 0$, then $\underline{\Phi}$ is a conservative vector field.
4. Once again, Tardy Terry arrived late for work. She claims that she oversleeps a lot because she gets tired out from sleepwalking. Last night, a howling wind came through an open window with force $\underline{F} =$

4. (continued)

$(2xye^x + x^2ye^x)\underline{i} + x^2e^x\underline{j}$ and swept her through the house while she sleepwalked.

(a) Is the work done by the wind independent of path? Explain.

(b) 

At one time last night, Tardy Terry walked the path shown at the left. Each circle of radius one was traversed once in the specified direction. Compute the line integral along the path from $(-2,1)$ to $(4,1)$.

ANSWERS TO PREREQUISITE QUIZ

1. (a) $10/3$

(b) 3

2. $-3y\underline{i} - 3x\underline{j}$

3. $\partial f/\partial x = 6xyz^{1/2}$; $\partial f/\partial y = 3x^2z^{1/2}$; $\partial f/\partial z = 3x^2y/2z^{1/2}$

ANSWERS TO SECTION QUIZ

1. $2\cos(1)$; orientation isn't necessary here since the line integral is independent of path.

2. (a) $x^3y/3 + x\cos y + C$

(b) $3x + e^x - y^2/2 + C$

(c) Not conservative

(d) Not conservative; vector field is undefined if $y = 0$.

3. False; let $\underline{\Phi} = y\underline{i}$ and let C be the line segment from $(0,0)$ to $(0,1)$ to $(0,0)$

4. (a) Yes; the cross-derivative test is satisfied.

(b) $16e^4 - 4e^{-2}$

18.3 Exact Differentials

PREREQUISITES

1. Recall the cross-derivative test (Section 18.2).
2. Recall how to solve separable differential equations in one variable (Section 8.5).

PREREQUISITE QUIZ

1. State the cross-derivative test.
2. Is $\underline{\Phi} = xy\underline{i} + xy\underline{j}$ conservative? Explain.
3. Solve the differential equation $dy/dx = (x^2 + x)/y$.

GOALS

1. Be able to solve exact differential equations.
2. Be able to use integrating factors to transform differential equations into exact ones.

STUDY HINTS

1. Cross-derivative test. Only the notation is different from the test presented in Section 18.2. P replaces $a(x,y)$ and Q replaces $b(x,y)$. Again, if the mixed partial derivatives are equal, the vector field is a gradient. If they are not equal, no antiderivative exists.
2. Finding an antiderivative. Yet another method for finding antiderivatives is discussed in method 2 of Example 1. Another method was given in Example 5 of Section 18.2, and another one was discussed in the study hints of the last section. All of these methods yield the same results. Learn the one that is easiest for you.

3. Exact differentials. This is just another term to describe an integrand which has previously been described by the words "conservative" and "gradient".

4. Exact differential equations. $P + Q(dy/dx) = 0$ can be multiplied by dx to get $Pdx + Qdy = 0$. If $Pdx + Qdy$ is exact, and $P = f_x$ and $Q = f_y$, then $f(x,y) = C$ is the solution of $P + Q(dy/dx) = 0$. This is because $df = (\partial f/\partial x)dx + (\partial f/\partial y)dy$ and $df = 0$ means that f is constant.

5. Integrating factors. This is a device used to transform equations into exact ones. If we multiply $M + N(dy/dx) = 0$ by an integrating factor, μ, we get $\mu M + \mu N(dy/dx) = 0$. Taking partials, we want $(\mu M)_y = (\mu N)_x$. According to the text, this is possible if $(M_y - N_x)/N$ is a factor of x alone in which case, $\mu = \exp[\int((M_y - N_x)/N)dx]$ if μ is a function of x alone. It is unpleasant and impractical to memorize this. Instead, try multiplying by $\mu(x)$ and apply the cross-derivative test to find μ. If this fails, try multiplying by $\mu(y)$.

SOLUTIONS TO EVERY OTHER ODD EXERCISE

1. Let $P = 2x + (xy + 1)\exp(xy)$ and $Q = x^2\exp(xy)$. Then, $\partial P/\partial y = x\exp(xy) + (xy + 1)y\exp(xy)$ and $\partial Q/\partial x = 2x\exp(xy) + x^2(x)\exp(xy)$. Since $\partial P/\partial y \neq \partial Q/\partial x$, the function is not an exact differential.

5. Use the cross-derivative test. Let $P = x^2y$ and $Q = x^3y/3$. Then $\partial P/\partial y = x^2$ and $\partial Q/\partial x = 3x^2y/3 = x^2y$. Since $\partial P/\partial y \neq \partial Q/\partial x$, the expression is not exact.

9. The function f is the integral of $Pdx + Qdy$ along any path from $(0,0)$ to (x,y). $\hat{f}(x,y)$ is the integral along the path from $(0,0)$ to $(0,y)$ and then from $(0,y)$ to (x,y).

13. If $\partial P/\partial y = \partial Q/\partial x$ for the equation $P + Q(dy/dx) = 0$, then the differential equation is exact and can be solved by the method of Example 3. $P = ye^x + e^y$, so $\partial P/\partial y = e^x + e^y$; $Q = xe^y + e^x$, so $\partial Q/\partial x = e^y + e^x$. It is exact, so $f(x,y) = \int_0^x dt + \int_0^y (xe^t + e^x)dt = t|_0^x + (xe^t + te^x)|_0^y = x + xe^y + ye^x - x = xe^y + ye^x = C$. $y(0) = 2$ implies $C = 2$; therefore, the solution is $xe^y + ye^x = 2$.

17. $P + Q(dy/dx) = 0$ is exact if $\partial P/\partial y = \partial Q/\partial x$. Then, the equation is solved by the method of Worked Example 3. $P = y + x^2y^2 - 1$, so $\partial P/\partial y = 1 + x^2$; $Q = x$, so $\partial Q/\partial x = 1$. The differential equation is not exact.

21. Use the method of Example 4.

(a) $\partial P/\partial y = -2x(2y)/(y^2 + 1)^2 = -4xy/(y^2 + 1)^2$ and $\partial Q/\partial x = -4xy/(y^2 + 1)^2$. The integrand is exact, so we may reparametrize the curve from $(-1,0)$ to $(0,0)$. Let it be $(t - 1,0)$, $0 \leqslant t \leqslant 1$. Then the line integral is $\int_0^1 2(t - 1)dt = 2(t^2/2 - t)|_0^1 = -1$.

(b) Rearrangement of the integrand in (a) yields $x/y = [(x^2 + 1)/(y^2 + 1)]dy/dx$. The antiderivative of the integrand is $(x^2 + 1)/(y^2 + 1)$, so the solution of the differential equation is $(x^2 + 1)/(y^2 + 1) = C$. $y(1) = 1$ implies $C = 1$, so the solution is $(x^2 + 1)/(y^2 + 1) = 1$, i.e., $x^2 + 1 = y^2 + 1$, i.e., $x = y$. Since $y(1) = 1$, $y = -x$ is <u>not</u> a solution.

25. $\mu = \exp[\int [(M_y - N_x)/N]dx]$. $M = 2y \cos y + x$, so $M_y = 2 \cos y - 2y \sin y$; $N = x \cos y - xy \sin y$, so $N_x = \cos y - y \sin y$. Thus, $(M_y - N_x)/N = (\cos y - y \sin y)/x(\cos y - y \sin y) = 1/x$. Therefore, $\mu = \exp(\int (1/x)dx) = \exp(\ln(x)) = x$.

29. If μ is a function of y alone, then $\mu_y = \mu'(y)$ and $\mu_x = 0$. Then, $\mu_y M + \mu M_y = \mu_x N + \mu N_x$ becomes $\mu_y M = \mu(N_x - M_y)$ or $\mu'/\mu = (N_x - M_y)/M$, i.e., $\ln \mu = \int [(N_x - M_y)/M]\,dy$.

SECTION QUIZ

1. Determine which of the following differential equations is exact, and solve the ones which are exact.

   (a) $(5xy + 3 + \sin y) + (5x^2/2 + x \cos y)dy/dx = 0$; $y(0) = 2$.

   (b) $(\sin xy + 2x + 5y) + (\sin xy + 2x + 5y)dy/dx = 0$; $y(0) = 1$.

   (c) $(ye^x - \sinh y) + (e^x - x \cosh y)y' = 0$; $y(0) = -1$.

2. Use the method of integrating factors to solve the differential equation $x^2y + x^3(dy/dx) = 0$.

3. At Andy the Anteater's favorite anthole, only the fastest ants dare to venture away from the hole. Andy's tongue, known as the fastest in the West, flicks out at a speed v, which is dependent upon his position r from the anthole. It is known that v and r are related by $v^2 + ve^{vr} + (2rv + re^{vr})(dv/dr) = 0$. Knowing that $v(0) = 4$, solve the differential equation.

ANSWERS TO PREREQUISITE QUIZ

1. A vector field $\underline{\Phi}(x,y) = a(x,y)\underline{i} + b(x,y)\underline{j}$ is conservative if and only if $\partial a/\partial y = \partial b/\partial x$.

2. No; $a_y = x \neq y = b_x$.

3. $y^2/2 = x^3/3 + x^2/2 + C$

ANSWERS TO SECTION QUIZ

1. (a) $5x^2y/2 + x \sin y + 3x + 2 = 0$

   (b) Not exact

   (c) $ye^x - x \sinh y + 1 = 0$

2. $xy = C$

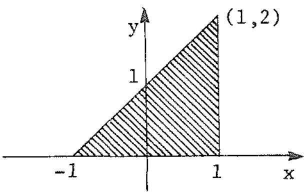
3. $rv^2 + e^{vr} = 1$

18.4 Green's Theorem

PREREQUISITES

1. Recall how to compute line integrals (Section 18.1).
2. Recall how to perform double integration (Sections 17.1, 17.2, and 17.3).
3. Recall how to parametrize a curve (Sections 2.4 and 10.4).

PREREQUISITE QUIZ

1. If $\underline{\Phi} = z\underline{i} - 2yx\underline{j}$ and C is the following curve: $(x,y,z) = (1-t,t,t^2)$, $0 \leqslant t \leqslant 2$, what is $\int_C \underline{\Phi}\, dt$?
2. Compute $\iint_D \cos y\, dx\, dy$ where D is the square $[0,\pi/2] \times [\pi,3\pi/2]$.
3. 

Write the area of the triangle as a double integral.
4. Given that $x = a \sin bt$, $0 \leqslant t \leqslant 2\pi$; find the values for a and b and find an expression for y such $x^2 + y^2 = 4$.

GOALS

1. Be able to convert certain line integrals to double integrals and vice versa by using Green's theorem.
2. Be able to calculate areas by using Green's theorem.

STUDY HINTS

1. Green's theorem. This theorem allows you to convert line integrals to double integrals, which may be easier to compute. You should memorize the result $\int_C (P dx + Q dy) = \iint_D (\partial Q/\partial x - \partial P/\partial y) dx\, dy$, where C is the

1. (continued)

boundary of D. Example 4 shows one of the advantages of Green's theorem.

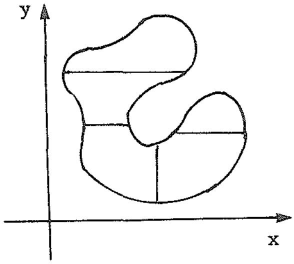
2. Key points. Notice that C is closed and must be traversed in a counterclockwise direction. Also continuous partial derivatives are required. Example 1, Section 18.5 shows what happens if the conditions are not met. If Green's theorem does not apply directly to a region, the region may be subdivided so that the theorem can be applied. See Example 2.

3. Area. From Green's theorem, we get $A = (1/2)\int_C (xdy - ydx)$. In most cases, you will compute areas by using normal methods of integration. However, if the curve bounding the region is given parametrically, Green's theorem can be much easier; a case is Exercise 24, p. 913.

SOLUTIONS TO EVERY OTHER ODD EXERCISE

1. We want to show that $\int_C (Pdx + Qdy) = \iint_D (\partial Q/\partial x - \partial P/\partial y)dx\, dy$, where C is traversed counterclockwise around D. On the left-hand side, we have $\int_C (Pdx + Qdy) = \int_C (xy\, dx + xdy)$. x and x^2 intersect at $x = 0$ and $x = 1$, so let C_1 be (t,t^2) for $0 \leqslant t \leqslant 1$ and let C_2 be $(1 - t, 1 - t)$ for $0 \leqslant t \leqslant 1$. Thus, the line integral is $\int_0^1 [(t)(t^2)(dt) + t(2t\, dt)] + \int_0^1 [(1 - t)(1 - t)(-dt) + (1 - t)(dt)] = \int_0^1 (t^3 + 2t^2 - (1 - 2t + t^2) - (1 - t))dt = \int_0^1 (t^3 + t^2 + 3t - 2)dt = (t^4/4 + t^3/3 + 3t^2/2 - 2t)\Big|_0^1 = 1/12$.

The right-hand side is $\iint_D (\partial Q/\partial x - \partial P/\partial y)dx\, dy = \int_0^1 \int_{x^2}^{x} (1 - x)dy\, dx = \int_0^1 [(y - xy)\Big|_{y=x^2}^{x}]dx = \int_0^1 (x^3 - 2x^2 + x)dx = (x^4/4 - 2x^3/3 + x^2/2)\Big|_0^1 = 1/12$.

5. 

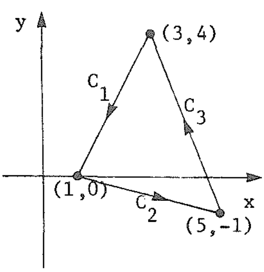
The sketch at the left shows only one way of dividing the region into type 1 and type 2 subregions.

9. Let $P = xy^2$ and $Q = -yx^2$; thus, $\partial Q/\partial x = -2xy$ and $\partial P/\partial y = 2xy$. By Green's theorem, $\int_C \underline{\Phi}(\underline{r})\cdot d\underline{r} = \iint_D - 4xy\, dx\, dy =$ $-4\int_{-b}^{b}\int_{-(a/b)\sqrt{b^2-y^2}}^{(a/b)\sqrt{b^2-y^2}} xy\, dx\, dy = -4\int_{-b}^{b}[(x^2y/2)\Big|_{x=-(a/b)\sqrt{b^2-y^2}}^{(a/b)\sqrt{b^2-y^2}}]\, dy =$ $-4\int_{-b}^{b} 0\, dy = 0$.

13. Using Green's theorem, we have $\int_C(P\, dx + Q\, dy) = \iint_D(\partial Q/\partial x - \partial P/\partial y)dx\, dy =$ $\int_1^3\int_2^3(\partial Q/\partial x - \partial P/\partial y)dy\, dx$. Here, $\partial Q/\partial x = 0$ and $\partial P/\partial y = 4y$, so the integral is $\int_1^3\int_2^3(-4y)dy\, dx = \int_1^3(-2y^2)\Big|_{y=2}^{3}dx = -10\int_1^3 dx = -20$.

17. (a) Recall that a zero dot product implies that the vectors are orthogonal. $(P\underline{i} + Q\underline{j})\cdot(Q\underline{i} - P\underline{j}) = PQ - QP = 0$.

(b) By Green's theorem, we have $\iint_D(\partial P/\partial x - \partial(-Q)/\partial y)dx\, dy =$ $\int_C[(-Q)dx + P\, dy]$. Given some parametrization of C : $x = x(t)$, $y = y(t)$, we know that $dx = kP$ and $dy = kQ$, where k is a constant, because $P\underline{i} + Q\underline{j}$ is parallel to the tangent vector. Therefore, the line integral is $\int_C[(-Q)(kP) + P(kQ)]dt = \int_C 0\, dt =$ 0 .

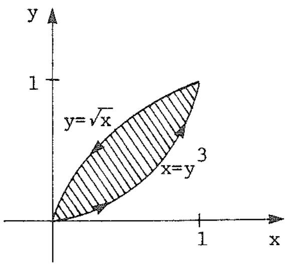
21. 

Let C be composed of C_1 , C_2 , and C_3 as shown at the left. For $0 \leqslant t \leqslant 1$, C_1 is $(1 - t)(3,4) + t(1,0) = (3 - 2t, 4 - 4t)$, C_2 is $(1 - t)(1,0) + t(5,-1) = (1 + 4t,-t)$, and C_3 is $(1 - t)(5,-1) + t(3,4) = (5 - 2t,-1 + 5t)$. By the corollary,

21. (continued)

the area of the triangle is $A = (1/2)\int_C(x\,dy - y\,dx) =$ $(1/2)\{\int_0^1[(3 - 2t)(-4dt) - (4 - 4t)(-2dt)] + \int_0^1[(1 + 4t)(-dt) - (-t)(4dt)] + \int_0^1[(5 - 2t)(5dt) - (-1 + 5t)(-2dt)]\} = (1/2)\int_0^1 18\,dt = 9$.

25.



By Green's theorem, the area is $A =$ $(1/2)\int_C(x\,dy - y\,dx)$. Use the parametrization (t,t^3) , $0 \leqslant t \leqslant 1$ and $(1 - t,\sqrt{1 - t})$, $0 \leqslant t \leqslant 1$. Thus, $A =$ $(1/2)\{\int_0^1[(t)(3t^2dt) - (t^3)(dt)] + \int_0^1[(1 - t)(-dt/2\sqrt{1 - t}) - (\sqrt{1 - t})(-dt)]\} =$ $(1/2)\int_0^1(2t^3 + \sqrt{1 - t}/2)dt = (1/2)[t^4/2 - (1 - t)^{3/2}/3]\big|_0^1 = (1/2)(1/2 + (1/3) = 5/12$.

29. $\partial P/\partial y = 1 = \partial Q/\partial x$, so $P\,dx + Q\,dy$ is an exact differential; therefore, it is conservative and the line integral is independent of path. A closed curve begins and ends at the same point, so $\int_C(P\,dx + Q\,dy) = 0$.

33. As is described in most encyclopedias, a planimeter is a mechanical device for measuring the area of a region. The device is run around the boundary of the region while a simple, but a clever mechanism actually performs the integration $(1/2)\int_{\partial D}(x\,dy - y\,dx)$ mechanically, during the circuit. This integral is the area of the region, by Green's theorem.

SECTION QUIZ

1. Use Green's theorem to compute the following line integrals:

(a) $\int_C (5xy^2\,dx + y \sin x\,dy)$, where C is the path along the square $[0,\pi] \times [0,\pi]$, traversed counterclockwise.

(b) $\int_C [(5x^2 + 6y^3)dx + (e^x + 4xy)dy]$, where C is the triangular path from $(0,0)$ to $(0,1)$ to $(1,0)$ to $(0,0)$.

2. Use Green's theorem to find the area of the region bounded by the parametrized curve (t,t^2), $-1 \leqslant t \leqslant 2$ and the line segment joining the endpoints.

3. A bearded old man, dressed in rags, wishes to know about his future. A fortune teller informs him that the crystal ball shows a harem of dancing girls inside an odd shaped room bounded by the ellipse $x^2 + y^2/4 = 1$.
   (a) Give a parametrization for the boundary of the room.
   (b) Use the corollary to Green's theorem to calculate the area of the room.
   (c) If the average harem girl occupies $(1/8)$ square units, how many girls can the dirty (according to his clothing) old man keep?

ANSWERS TO PREREQUISITE QUIZ

1. $-4/3$
2. -2π
3. $\int_{-1}^{1}\int_{0}^{x+1} dy\,dx$
4. $a = 2$; $b = 1$; $y = 2\cos t$

ANSWERS TO SECTION QUIZ

1. (a) $-5\pi^4/2$
   (b) $17/6 - e$ (Note the orientation of C.)
2. $9/2$
3. (a) $(\cos t, 2\sin t)$, $0 \leqslant t \leqslant 2\pi$
   (b) 2π
   (c) 16π or about 49 girls

18.5 Circulation and Stokes' Theorem

PREREQUISITES

1. Recall how to use Green's theorem (Section 18.4).
2. Recall how to compute a cross product (Section 13.5).

PREREQUISITE QUIZ

1. When using Green's theorem, in what direction must the boundary be traversed?
2. Fill in the blanks: Green's theorem states that $\int_C$ __________ $=$ $\iint_D$ __________ .
3. Use Green's theorem to evaluate $\int_C [(x^2y + y)dx + (3 \cos x)dy]$, where C is the rectangle $[0,1] \times [2,3]$ traversed clockwise.
4. Evaluate $(3\underline{i} + 2\underline{j} + \underline{k}) \times (-\underline{i} - 2\underline{j} + 2\underline{k})$.

GOALS

1. Be able to compute the curl of a vector field and explain its physical interpretation.
2. Be able to convert a line integral into a surface integral by using Stokes' theorem.

STUDY HINTS

1. Scalar curl. For $\underline{\Phi} = P\underline{i} + Q\underline{j}$, the scalar curl is defined to be $\partial Q/\partial x - \partial P/\partial y$. This is the integrand of the double integral in Green's theorem. Physically, the scalar curl tells you about the circulation per unit area. Circulation is a term which describes the rotational motion of fluids, such as tea circulating in a cup or air circulating in a room.

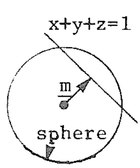
2. Surface integral. This is defined as $\iint_S \underline{\Phi}\cdot\underline{n}dA = \iint_S \underline{\Phi}\cdot d\underline{A} = \iint_D (P\underline{i} + Q\underline{j} + R\underline{k})\cdot(-f_x\underline{i} - f_y\underline{j} + \underline{k})dx\,dy$, if S is the graph of f .

3. Curl. It is defined by curl $\underline{\Phi} = (R_y - Q_z)\underline{i} + (P_z - R_x)\underline{j} + (Q_x - P_y)\underline{k}$. You should remember this by curl $\underline{\Phi} = \underline{\nabla} \times \underline{\Phi}$, where $\underline{\nabla} = (\partial/\partial x)\underline{i} + (\partial/\partial y)\underline{j} + (\partial/\partial z)\underline{k}$.

4. Stokes' theorem. Green's theorem was restricted to the xy-plane. Stokes' theorem, like Green's, allows you to convert a line integral to a double integral. It is more general in that the boundary, ∂S , and the region of integration, S , are no longer restricted to a plane. Again, the boundary must be traversed counterclockwise and continuous differentiability is required. By "counterclockwise," we mean that if you walk around the boundary, the region is on your left (See Fig. 18.5.4). The formula is $\int_{\partial S}\underline{\Phi}(\underline{r})\cdot d\underline{r} = \iint_S (\underline{\nabla} \times \underline{\Phi})\cdot\underline{n}dA$. Note that if $\underline{\Phi} = P(x,y)\underline{i} + Q(x,y)\underline{j}$ and S is a plane region, you get Green's theorem. (Some students like to use this method to remember Green's theorem.)

5. Applications. Method 2 of Example 7 shows one of the uses of Stokes' theorem. According to the theorem, we can change a surface to any other surface with the same boundary. In most cases, we will change to a planar surface.

6. Example 7 clarified. Some people have said that $\underline{m}$ stands for "mysterious" in Method 1 of Example 7.



Looking at the plane $x + y + z = 1$ edge on, we see that it is a circle that is cut out and that the vector required is orthogonal to the plane with length equal to the distance of the plane from the origin. Thus, $\underline{m} = a(\underline{i} + \underline{j} + \underline{k})$. Using the distance from a point

7. (continued)

to a plane equation (formula (7), p. 674), we get $\|\underline{m}\| = 1/\sqrt{3}$, so $a = 1/3$. Therefore, $\underline{m} = (1/3)(\underline{i} + \underline{j} + \underline{k})$.

SOLUTIONS TO EVERY OTHER ODD EXERCISE

1. $\partial Q/\partial x - \partial P/\partial y$ is the scalar curl of the vector field $P\underline{i} + Q\underline{j}$. $\partial Q/\partial x = -1$ and $\partial P/\partial y = 1$, so the scalar curl is -2.

5. The surface integral of $\underline{\Phi}$ over S, where $\underline{\Phi} = P\underline{i} + Q\underline{j} + R\underline{k}$ and S is the surface $z = f(x,y)$, is $\iint_D(-Pf_x - Qf_y + R)dx\,dy$. $z = 2x - y$, so $f_x = 2$ and $f_y = 1$; $P = 3x^2$, $Q = -2yx$, and $R = 8$. Thus, the surface integral is $\int_0^2\int_0^2(-6x^2 + 2xy + 8)dx\,dy = \int_0^2(-2x^3 + x^2y + 8x)\big|_{x=0}^2 dy = \int_0^2 4y\,dy = 2y^2\big|_0^2 = 8$.

9. The curl of $\underline{\Phi} = P\underline{i} + Q\underline{j} + R\underline{k}$ is $(R_y - Q_z)\underline{i} + (P_z - R_x)\underline{j} + (Q_x - P_y)\underline{k} = \underline{\nabla} \times \underline{\Phi}$. In this case, $\underline{\nabla} \times \underline{F} = \begin{vmatrix} \underline{i} & \underline{j} & \underline{k} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ e^z & -\cos(xy) & z^3y \end{vmatrix} = z^3\underline{i} + e^z\underline{j} + (y \sin xy)\underline{k}$.

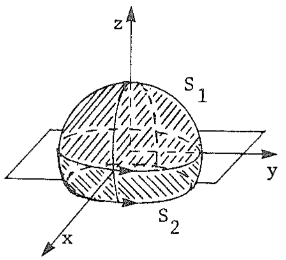
13. $\mathrm{Curl}\ (f\underline{\Phi}) = \begin{vmatrix} \underline{i} & \underline{j} & \underline{k} \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ fP & fQ & fR \end{vmatrix}$, where $\underline{\Phi} = P\underline{i} + Q\underline{j} + R\underline{k}$. This is $(\partial fR/\partial y - \partial fQ/\partial z)\underline{i} + (\partial fP/\partial z - \partial fR/\partial x)\underline{j} + (\partial fQ/\partial x - \partial fP/\partial y)\underline{k}$. By the product rule for differentiation, we get $[(\partial f/\partial y)R + f(\partial R/\partial y) - (\partial f/\partial z)Q - f(\partial Q/\partial z)]\underline{i} + [(\partial f/\partial z)P + f(\partial P/\partial z) - (\partial f/\partial x)R - f(\partial R/\partial x)]\underline{j} + [(\partial f/\partial x)Q + f(\partial Q/\partial x) - (\partial f/\partial y)P - f(\partial P/\partial y)]\underline{k} = f[(R_y - Q_z)\underline{i} + (P_z - R_x)\underline{j} + (Q_x - P_y)\underline{k}] + [(\partial f/\partial y)R - (\partial f/\partial z)Q]\underline{i} + [(\partial f/\partial z)P - (\partial f/\partial x)R]\underline{j} + [(\partial f/\partial x)Q - (\partial f/\partial y)P]\underline{k}$. We recognize the first half as f curl $\underline{\Phi}$ and the second half is $\underline{\nabla} f \times \underline{\Phi}$.

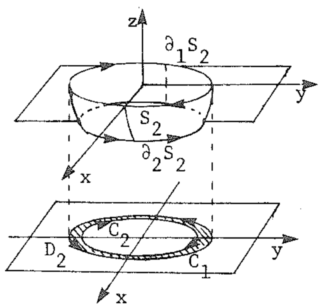
17. $\underline{\nabla} \times \underline{\Phi} = \begin{vmatrix} \underline{i} & \underline{j} & \underline{k} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ P & Q & Q \end{vmatrix}$, where $P = 1/(y + z)$ and $Q = -x/(y + z)^2$. By symmetry, note that $\partial Q/\partial y = \partial Q/\partial z$ and $\partial P/\partial y = \partial P/\partial z$. Thus, $\underline{\nabla} \times \underline{\Phi}$ reduces to $(\partial Q/\partial x - \partial P/\partial y)\underline{k}$. However, $\partial Q/\partial x = -1/(y + z)^2$ and $\partial P/\partial y = -1/(y + z)^2$; therefore, $\underline{\nabla} \times \underline{\Phi} = \underline{0}$. By Stokes' theorem, $\int_C \underline{\Phi}(\underline{r})\cdot d\underline{r} = \iint_S (\underline{\nabla} \times \underline{\Phi})\cdot \underline{n} dA = \iint_S \underline{0}\cdot\underline{n}\, dA = 0$.

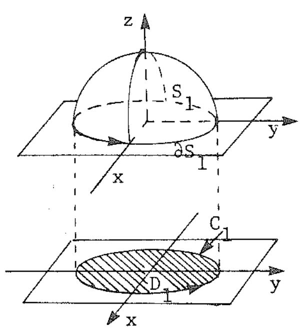
21. Let ∂S be parametrized by $\underline{\sigma}(t)$. By Stokes' theorem, $\iint_S (\underline{\nabla} \times \underline{\Phi})\cdot \underline{n}\, dA = \int_{\partial S} \underline{\Phi} = \int_{\underline{\sigma}(t)} \underline{\Phi} = \int \underline{\Phi}(\underline{\sigma}(t))\cdot\underline{\sigma}'(t)dt = 0$ since $\underline{\Phi}(\underline{\sigma}(t))$ is perpendicular to $\underline{\sigma}'(t)$.

25. In symbols, the circulation of $\underline{H}$ around C is $\int_C \underline{H}(\underline{r})\cdot d\underline{r}$. This is equal to the flux of $\underline{J}$ across S , which is $\iint_S \underline{J}\cdot\underline{n}\, dA$. Starting with $\underline{\nabla} \times \underline{H} = \underline{J}$, we take the scalar product of both sides with the outward pointing normal $\underline{n}$ to get $(\underline{\nabla} \times \underline{H})\cdot\underline{n} = \underline{J}\cdot\underline{n}$. Integrate over S to get $\iint_S (\underline{\nabla} \times \underline{H})\cdot\underline{n}\, dA = \iint_S \underline{J}\cdot\underline{n}\, dA$. Then, use Stokes' theorem to get $\int_C \underline{H}(\underline{r})\cdot d\underline{r} = \iint_S \underline{J}\cdot\underline{n}\, dA$, which is Ampere's law.

29.







The surface S is shaded in the figure at the left. It consists of two parts, drawn separately in the figures below. If $\underline{\Phi}$ is any continuously differentiable vector field in space, then Stokes' theorem as written in the box on p. 918 applies directly to the hemisphere S_1 to give $\int_{\partial S_1} \underline{\Phi}(\underline{r})\cdot d\underline{r} = \iint_{S_1} (\underline{\nabla} \times \underline{\Phi})\cdot \underline{n} dA$.

Now, we may also apply Stokes' theorem to the surface S_2, but a little extra care is necessary. The orientation of S_2 as the graph of a function on the region D_2 (normal pointing upward) is the opposite of its orientation as a part of the surface S (normal pointing outward), so Stokes' theorem in this case gives us (since the boundary of D_2 is $C_1 - C_2$) $-\int_{\partial_1 S_2} \underline{\Phi}(\underline{r})\cdot d\underline{r} - \int_{\partial_2 S_2} \underline{\Phi}(\underline{r})\cdot d\underline{r} = -\iint_{S_2} (\underline{\nabla} \times \underline{\Phi})\cdot \underline{n}\, dA$. Subtracting the second equation above from the first, and using the facts that $S = S_1 + S_2$, $\partial S_1 = -\partial_1 S_2$, and $\partial_2 S_2 = \partial S$, we obtain the result $\int_{\partial S} \underline{\Phi}(\underline{r})\cdot d\underline{r} = \iint_S (\underline{\nabla} \times \underline{\Phi})\cdot \underline{n}\, dA$, which is Stokes' theorem for the region S.

SECTION QUIZ

1. Calculate the surface integral of $\underline{\Phi} = x\underline{j} + 2z\underline{k}$ over the surface S, where S is defined by $z = xy^2$ over the triangle bounded by $x = 0$, $y = 0$, and $y = x - 1$.

2. Let $\underline{\Phi} = 2xe^y\underline{i} + (x^2e^y + z^3)\underline{j} + 3z^2y\underline{k}$. Use Stokes' theorem to compute $\int_C \underline{\Phi}(\underline{r})\cdot d\underline{r}$, where C is the boundary of the semi-ellipsoid $x^2 + 4y^2 + 9z^2 = 25$, $z \geqslant 0$.

3. Find the scalar curl of $\underline{W}(r,s) = (s \ln r + r^2 + s)(\underline{i} + \underline{j}) + r\underline{j}$.

4. On a calm, sunny day, an old fisherman got his line caught. Unknown to him, the line was caught on the rusty old lock of an old sea fortress. Tugging with all of his might, he yanked the lock off, forming an odd-looking whirlpool. The velocity field of the water is given by $\underline{V} = (x + y)^2\underline{i} + xy\underline{j} - z\underline{k}$.

   (a) Compute curl $\underline{V}$.

   (b) The circulation is defined as $\int_C \underline{V}(x,y,z)\cdot(dx\,\underline{i} + dy\,\underline{j} + dz\,\underline{k})$. If C is the boundary of a surface S and $\underline{n}$ is normal to S, write the circulation as a surface integral.

ANSWERS TO PREREQUISITE QUIZ

1. Counterclockwise

2. $\int_C(P\,dx + Q\,dy) = \iint_D(\partial Q/\partial x - \partial P/\partial y)\,dx\,dy$

3. $-3\cos(1) + 13/3$

4. $6\underline{i} - 7\underline{j} - 4\underline{k}$

ANSWERS TO SECTION QUIZ

1. $1/15$

2. 0

3. $s/r + 2r - \ln r$

4. (a) $-(2x + y)\underline{k}$

   (b) $-\iint_S [(2x + y)\underline{k}] \cdot \underline{n}\, dA$

18.6 Flux and the Divergence Theorem

PREREQUISITES

1. Recall how to compute a dot product (Section 13.4).
2. Recall how to compute a surface integral (Section 18.5).

PREREQUISITE QUIZ

1. Evalaute $(3\underline{i} + 2\underline{j} + \underline{k})\cdot(-\underline{i} - 2\underline{j} + 2\underline{k})$.
2. If $\underline{\Phi} = \underline{i} + \underline{j} + \underline{k}$ is a vector field in space and S is the surface $f(x,y) = x^2 + y^2$ over the rectangle $[0,1] \times [0,2]$, what is the surface integral $\iint_S \underline{\Phi}\cdot\underline{n}\, dA$?

GOALS

1. Be able to compute the divergence of a vector field and explain its physical interpretation.
2. Be able to compute the flux of a vector field by using Gauss' divergence theorem.

STUDY HINTS

1. Flux. The flux of $\underline{V}$ across C is the volume of substance (usually air or water) crossing C per unit time as the substance moves with velocity $\underline{V}$. It is defined by $\int_C \underline{V}\cdot\underline{n}\, ds = \int_C (P dy - Q dx)$.
2. Divergence. This is defined by $\operatorname{div} \underline{V} = \underline{\nabla}\cdot\underline{V}$, where $\underline{\nabla} = (\partial/\partial x)\underline{i} + (\partial/\partial y)\underline{j} + (\partial/\partial z)\underline{k}$. Physically, if $\operatorname{div} \underline{V} < 0$, a fluid is compressible, i.e., fluid is squeezing. If $\operatorname{div} \underline{V} > 0$, the fluid is expanding or diverging. If $\operatorname{div} \underline{V} = 0$, the fluid is incompressible or divergence free.
3. Gauss' theorem in the plane. This theorem states that flux $= \int_C \underline{V}\cdot\underline{n}\, ds = \iint_D (\operatorname{div} \underline{V}) dx\, dy$. Again, the conversion is made from line integral to

3. (continued)

double integral. Counterclockwise orientation and continuous differentiability remain requirements.

4. Gauss' theorem in space. This theorem states that flux in space is $\iint_{\partial W}(\underline{V}\cdot\underline{n})\,dA = \iiint_W(\operatorname{div}\underline{V})\,dx\,dy\,dz$. Here, the conversion is made from a double to a triple integral.

SOLUTIONS TO EVERY OTHER ODD EXERCISE

1. The divergence of $P\underline{i} + Q\underline{j}$ is $\partial P/\partial x + \partial Q/\partial y$, so $\operatorname{div}\underline{\Phi} = 3x^2 - x^2\cos(xy)$.

5. The flux of $\underline{\Phi}$ across the perimeter of C is $\int_C \underline{\Phi}\cdot\underline{n}\,ds = \iint_D(\operatorname{div}\underline{\Phi})\,dx\,dy$. In this case, the flux is $\int_{-1}^{1}\int_{-1}^{1}(2x - 3y^2)\,dx\,dy = \int_{-1}^{1}(x^2 - 3xy^2)\Big|_{x=-1}^{1}dy = \int_{-1}^{1} -6y^2\,dy = -2y^3\Big|_{-1}^{1} = -4$.

9. $\operatorname{div}\underline{V} > 0$ if the fluid appears to be emerging from smaller regions. $\operatorname{div}\underline{V} < 0$ if the fluid is converging.

It appears that $\operatorname{div}\underline{V} > 0$ at A and C ; $\operatorname{div}\underline{V} < 0$ at B and D .

13. By definition, $\operatorname{div}\underline{V} = \partial P/\partial x + \partial Q/\partial y + \partial R/\partial z$, where $\underline{V} = P\underline{i} + Q\underline{j} + R\underline{k}$. Here, $\operatorname{div}\underline{V} = 1 + 1 + 1 = 3$.

17. By direct calculation, we must compute $\iint_{\partial W}\underline{F}\cdot\underline{n}\,dA$ for each of the six faces. When $x = 0$, we have $0 \leqslant y \leqslant 1$, $0 \leqslant z \leqslant 1$ and $\underline{n} = -\underline{i}$. Similarly when $x = 1$, we have $0 \leqslant y \leqslant 1$, $0 \leqslant z \leqslant 1$ and $\underline{n} = \underline{i}$. Thus, when $x = 0$, $\iint_{\partial W}\underline{F}\cdot\underline{n}\,dA = \int_0^1\int_0^1 -x\,dy\,dz = \int_0^1\int_0^1 0\,dy\,dz = 0$, and when $x = 1$, $\iint_{\partial W}\underline{F}\cdot\underline{n}\,dA = \int_0^1\int_0^1 x\,dy\,dz = \int_0^1\int_0^1 1\,dy\,dz = 1$.

When y is constant, the faces may be described by $0 \leqslant x \leqslant 1$ and $0 \leqslant z \leqslant 1$. $\underline{n} = \underline{j}$ when $y = 1$ and $\underline{n} = -\underline{j}$ when $y = 0$. For $y = 0$,

17. (continued)

$\iint_{\partial W} \underline{F}\cdot\underline{n}\, dA = \int_0^1\int_0^1 -y\, dx\, dz = \int_0^1\int_0^1 0\, dx\, dz = 0$. When $y = 1$, $\iint_{\partial W} \underline{F}\cdot\underline{n}\, dA = \int_0^1\int_0^1 y\, dx\, dz = \int_0^1\int_0^1 1\, dx\, dz = 1$.

When z is held constant, we have $0 \leqslant x \leqslant 1$ and $0 \leqslant y \leqslant 1$. $\underline{n} = \underline{k}$ for $z = 1$ and $\underline{n} = -\underline{k}$ for $z = 0$. Therefore, when $z = 0$, $\iint_{\partial W} \underline{F}\cdot\underline{n}\, dA = \int_0^1\int_0^1 z\, dx\, dy = \int_0^1\int_0^1 0\, dx\, dy = 0$. When $z = 1$, $\iint_{\partial W} \underline{F}\cdot\underline{n}\, dA = \int_0^1\int_0^1 - z\, dx\, dy = \int_0^1\int_0^1 (-1)dx\, dy = -1$. Therefore, $\iint_{\partial W} \underline{F}\cdot\underline{n}\, dA$ over the unit cube is $0 + 1 + 0 + 1 + 0 + (-1) = 1$. According to the divergence theorem, $\iiint_W (\text{div } \underline{F}) dx\, dy\, dz$ should give the same answer. $\text{div } \underline{F} = 1 + 1 - 1 = 1$, so the triple integral is $\int_0^1\int_0^1\int_0^1 dx\, dy\, dz = 1$.

21. Rearrange to get $\iiint_W (\underline{\nabla} f)\cdot\underline{\Phi}\, dx\, dy\, dz + \iiint_W f\underline{\nabla}\cdot\underline{\Phi}\, dx\, dy\, dz = \iiint_W [(\underline{\nabla} f)\cdot\underline{\Phi} + f\underline{\nabla}\cdot\underline{\Phi}] dx\, dy\, dz = \iint_{\partial W} f\underline{\Phi}\cdot\underline{n}\, dA$. By Gauss' divergence theorem in space, this is true if $(\underline{\nabla} f)\cdot\underline{\Phi} + f\underline{\nabla}\cdot\underline{\Phi} = \text{div}(f\underline{\Phi})$. This is the identity proven in Example 6(a).

25. (a) By direct calculation or using Review Exercise 73(b), Chapter 15, one may verify that $\underline{\nabla}\cdot\underline{\nabla}(p(\underline{q})/4\pi\|\underline{p} - \underline{q}\|) = 0$ whenever $\underline{q} \neq \underline{p}$; here, differentiation is with respect to the vector variable $\underline{p}$. For any positive ε , let Ω_ε be the region obtained from Ω by deleting the ball B_ε of radius ε whose center is at $\underline{q}$. (We may choose ε small enough so that the ball is contained in Ω .) Applying the divergence theorem to $\underline{\nabla}(p(\underline{q})/4\pi\|\underline{p} - \underline{q}\|)$ on this region, with $\underline{p}$ as the variable of integration, gives $0 = \iint_{\partial\Omega_\varepsilon} \underline{\nabla}(\rho(\underline{q})/4\pi\|\underline{p} - \underline{q}\|)\cdot\underline{n}\, dA = \iint_{\partial\Omega} \underline{\nabla}(\rho(\underline{q})/4\pi\|\underline{p} - \underline{q}\|)\cdot\underline{n}\, dA - \iint_{\partial B_\varepsilon} \underline{\nabla}(\rho(\underline{q})/4\pi\|\underline{p} - \underline{q}\|)\cdot\underline{n}\, dA$. In the last integral, $\underline{\nabla}(\rho(\underline{q})/4\pi\|\underline{p} - \underline{q}\|)$ is equal to $-(\rho(\underline{q})/4\pi\varepsilon^2)\underline{n}$, because ∂B_ε is a sphere about $\underline{q}$. Since the area of ∂B_ε is $4\pi\varepsilon^2$, we get

25\. (a) (continued)

$\iint_{\partial\Omega} \underline{\nabla}(\rho(\underline{q})/4\pi\|\underline{p} - \underline{q}\|)\cdot\underline{n}\, dA = \rho(\underline{q})$. Integrating now with respect to $\underline{q}$ over the region Ω , reordering the integrations and the gradient operation on the left-hand side, and using the definition of ϕ in the statement of the exercise, we obtain $\iint_{\partial\Omega} \underline{\nabla}\phi\cdot\underline{n}\, dA = \iiint_{\Omega} \rho\, dx\, dy\, dz$.

(b) By the divergence theorem, $\iint_{\partial\Omega} \underline{\nabla}\phi\cdot\underline{n}\, dA = \iiint_{\Omega} \underline{\nabla}^2\phi\, dx\, dy\, dz$. Since this formula and the result of part (a) are true for arbitrary regions Ω , it follows that $\underline{\nabla}^2\phi = \rho$.

SECTION QUIZ

1\. Let $\underline{U} = xz\underline{i} - e^y \sin z\, \underline{j} + (z - \ln x)\underline{k}$.

(a) Calculate the divergence of $\underline{U}$.

(b) If $\underline{U}$ is the velocity field of a fluid, is the fluid compressing, incompressible, or expanding at $(1,1,0)$?

2\. (a) Express the flux of a vector field $\underline{B}$ across a surface S in terms of div $\underline{B}$.

(b) Calculate the flux of $xy\underline{i} + 3y\underline{j} + 2z\underline{k}$ through the boundary of the region in space defined by $0 \leqslant x \leqslant 2$, $0 \leqslant y \leqslant 3$, $0 \leqslant z \leqslant 2 + \sin x$.

3\. A hat collector didn't realize it was going to rain today, so he wore his favorite hat when he took his afternoon stroll. A sudden cloudburst damaged the hat and out of anger, the mad hatter blew his top (actually, he punched a lot of holes through his hat). The wind is making the rain's velocity field $\underline{V} = 2x\underline{i} + y\underline{j} - \underline{k}$. The hat can be described as the cylinder $x^2 + y^2 \leqslant 1$, $0 \leqslant z \leqslant 5$. Assuming that the mad hatter punched enough holes in his hat to allow the rain to pass freely, what is the flux of the rain water passing through the hat?

ANSWERS TO PREREQUISITE QUIZ

1. -5

2. -4

ANSWERS TO SECTION QUIZ

1. (a) $z - e^y \sin z + 1$

   (b) Expanding

2. (a) $\iiint_W (\mathrm{div}\ \underline{B})\, dx\, dy\, dz$, where S is the boundary of W .

   (b) $(39/2)(5 \cos(2))$

3. 10π

18.R Review Exercises for Chapter 18

SOLUTIONS TO EVERY OTHER ODD EXERCISE

1. Let the line be parametrized by $(2t, 1+t, 1-4t)$ for $0 \leqslant t \leqslant 1$, then $\int_C (xy\,dx + x \sin y\,dy) = \int_0^1 [2t(1+t)(2\,dt) + 2t\sin(1+t)dt] = \int_0^1 [4t + 4t^2 + 2t\sin(1+t)]dt = (2t^2 + 4t^3/3)\big|_0^1 + \int_0^1 2t\sin(1+t)dt$. Use integration by parts with $u = 2t$, $v = -\cos(1+t)$ to get $10/3 + 2t(-\cos(1+t))\big|_0^1 + 2\int_0^1 \cos(1+t)dt = 10/3 - 2\cos 2 + 2\sin(1+t)\big|_0^1 = 10/3 - 2\cos 2 + 2\sin 2 - 2\sin 1$.

5. We are integrating a gradient, so the integral is independent of path. $\int_C \underline{\nabla f} \cdot d\underline{r} = f(8,2,\pi) - f(0,1,0)$, where $f(x,y,z) = xy^2\cos z$. Thus, the integral is -32.

9. $\int_C (\sin x\,dx - \ln z\,dy + xy\,dz) = \int_1^2 [(\sin(2t+1))(2\,dt) + (\ln(t^2))(dt/t) + (2t+1)(\ln t)(2t\,dt)]$. $\int_1^2 2\sin(2t+1)dt = -\cos(2t+1)\big|_1^2 = -\cos 5 + \cos 3$. Let $u = \ln(t^2)$, so $du = 2\,dt/t$; therefore, $\int_1^2 (\ln(t^2))(dt/t) = \int_0^{\ln 4} u\,du/2 = (u^2/4)\big|_0^{\ln 4} = (\ln 4)^2/4$. Integrate $\int_1^2 (4t^2 + 2t)(\ln t)dt$ by parts with $u = \ln t$ and $v = 4t^3/3 + t^2$ to get $(4t^3/3 + t^2)(\ln t)\big|_1^2 - \int_1^2 (4t^2/3 + t)dt = (44\ln 2)/3 - (4t^3/9 + t^2/2)\big|_1^2 = (44\ln 2)/3 + 83/18$. Thus, the answer is $-\cos 5 + \cos 3 + (\ln 4)^2/4 + (44\ln 2)/3 + 83/18$.

13. We are integrating a gradient, so the integral is $f(\underline{\sigma}(t_2)) - f(\underline{\sigma}(t_1))$, where $f(x,y,z) = xze^y - z^3/(1+y^2)$. The integral is $f(\underline{\sigma}(1)) - f(\underline{\sigma}(0)) = f(1,0,1/3) - f(0,-1,1) = 1/3 - 1/27 + 1/2 = 43/54$.

17. Work is defined as the line integral $\int_C \underline{\Phi}(\underline{\sigma}(t)) \cdot \underline{\sigma}'(t)\,dt$. $\underline{\Phi}(\underline{\sigma}(t)) = e^t\underline{i} - t\exp(t^3)\underline{j}$ and $\underline{\sigma}'(t) = \underline{i} + 2t\underline{j}$, so the work done is $\int_0^1 (e^t - 2t^2\exp(t^3))dt$. Let $u = t^3$ to get $\int_0^1 e^t\,dt - 2\int_0^1 (e^u/3)du = e^t\big|_0^1 - (2e^u/3)\big|_0^1 = e/3 - 1/3$.

21. A vector field, $P\underline{i} + Q\underline{j}$, is conservative if and only if it is the gradient of some function. It is the gradient of some function if $\partial P/\partial y = \partial Q/\partial x$. Here, $\partial P/\partial y = 6x^2y$ and $\partial Q/\partial x = 6x^2y^2$, so $\underline{\Phi}(x,y)$ is not conservative.

25. $P\,dx + Q\,dy$ is an exact differential if $\partial P/\partial y = \partial Q/\partial x$. A potential function is constructed from $f(x,y) = \int_0^x a(t,0)dt + \int_0^y b(x,t)dt$ where $a(x,y) = P$ and $b(x,y) = Q$. The three variable case uses the result of Exercise 23. $\partial P/\partial y = e^y\sin x + xe^y\cos x$ and $\partial Q/\partial x = e^y\sin x + xe^y\cos x$, so this is an exact differential. An antiderivative is $f(x,y) = \int_0^x(\sin t + t\cos t)dt + \int_0^y xe^t \sin x\, dt = t\sin t|_0^x + xe^t\sin x|_0^y = x\sin x + xe^y\sin x - x\sin x + C = xe^y\sin x + C$.

29. For the differential equation, $P + Q(dy/dx) = 0$, the method of Example 1, Section 18.3 may be used if $\partial P/\partial y = \partial Q/\partial x$. $\partial P/\partial y = \cos x + 2xe^y$ and $\partial Q/\partial x = \cos x + 2xe^y$, so a solution exists. Thus, $f(x,y) = \int_0^x 2t\,dt + \int_0^y(\sin x + x^2e^t + 2)dt = t^2|_0^x + (t\sin x + x^2e^t + 2t)|_0^y = x^2 + y\sin x + x^2e^y + 2y - x^2 = y\sin x + x^2e^y + 2y = C$. $y(\pi/2) = 0$ implies $C = \pi^2/4$, so the solution is $y\sin x + x^2e^y + 2y = \pi^2/4$.

33. We have $\partial P/\partial y = x$ and $\partial Q/\partial x = 2x$. Since $\partial P/\partial y \neq \partial Q/\partial x$, the equation is not exact.

37. Use Green's theorem: $\int_C(P\,dx + Q\,dy) = \iint_D(\partial Q/\partial x - \partial P/\partial y)dx\,dy$.

(a) $P = 0 = \partial P/\partial y$ and $Q = x$, so $\partial Q/\partial x = 1$; therefore $\int_C x\,dy = \iint_D(1)\,dx\,dy$, which is the area of D .

(b) $P = y$, so $\partial P/\partial y = 1$ and $Q = 0 = \partial Q/\partial x$; therefore, $\int_C y\,dx = \iint_D(-1)\,dy\,dx$, which is the negative of the area of D .

(c) $P = x$, so $\partial P/\partial y = 0$ and $Q = 0 = \partial Q/\partial x$; therefore, $\int_C x\,dx = \iint_D 0\,dy\,dx = 0$.

41. The curl is $\underline{\nabla} \times \underline{\Phi}$ and the divergence is $\underline{\nabla} \cdot \underline{\Phi}$. We compute $\underline{\Phi} = 2xz\underline{j} - 2xy\underline{k}$, so curl $\underline{\Phi} = (-2x - 2x)\underline{i} + (-2y)\underline{j} + 2z\underline{k} = 4x\underline{i} - 2y\underline{j} + 2z\underline{k}$; div $\underline{\Phi} = 0$.

45. The work done by a force field $\underline{F}$ going around a closed curve C is $\int_C \underline{F}(r) \cdot d\underline{r}$. By Stokes' theorem, this is the same as the surface integral, $\iint_S (\underline{\nabla} \times \underline{F}) \cdot \underline{n}\, dA$, where C is the boundary of S.

49. $\iint_S \underline{F} \cdot \underline{n}\, dA = \iint_D (-Pf_x - Qf_y + R)dx\, dy$. $z = f(x,y) = (1 - x^2 - y^2)^{1/2}$ and D is the unit circle, so $f_x = -x/(1 - x^2 - y^2)^{1/2}$ and $f_y = -y/(1 - x^2 - y^2)^{1/2}$. The surface integral becomes, in polar coordinates, $\int_0^{2\pi}\int_0^1 [(x^2 + 3xy^5)/z + (y^2 + 10xyz)/z + (z - xy)]r\, dr\, d\theta$, where $x = r\cos\theta$, $y = r\sin\theta$, and $z = \sqrt{1 - r^2}$. This becomes $\int_0^{2\pi}\int_0^1 [(r^2 + 3r^6\cos\theta\sin^5\theta)/\sqrt{1 - r^2} + 9r^2\cos\theta\sin\theta + \sqrt{1 - r^2}]r\, dr\, d\theta$. $\int (r^3/\sqrt{1 - r^2})dr = -\int [(1 - u)/2\sqrt{u}]du = -\sqrt{u} + u^{3/2}/3 + C = -\sqrt{1 - r^2} + (1 - r^2)^{3/2}/3 + C$. $\int (r^7/\sqrt{1 - r^2})dr = -\int [(1 - u)^3/2\sqrt{u}]du = \int [(u^3 - 3u^2 + 3u - 1)/2\sqrt{u}]du = u^{7/2}/7 - 3u^{5/2}/5 + 3u^{3/2}/3 - u^{1/2} + C = (1 - r^2)^{7/2}/7 - 3(1 - r^2)^{5/2}/5 + (1 - r^2)^{3/2} - \sqrt{1 - r^2} + C$. $\int r^3 dr = r^4/4$; $\int r\sqrt{1 - r^2} = -(1 - r^2)^{3/2}/3 + C$. Therefore, after evaluating at 0 and 1, the integral reduces to $\int_0^{2\pi}(48\cos\theta\sin^5\theta/35 + 9\cos\theta\sin\theta/4 - 1/3)d\theta = (8\sin^6\theta/35 + 9\sin^2\theta/8 - \theta/3)\Big|_0^{2\pi} = -2\pi/3$.

53. Applying the divergence theorem to $\underline{\Phi}$ on a ball B_ε of radius ε about P_0 shows that the flux of $\underline{\Phi}$ through the boundary of B_ε is equal to $\iiint_{B_\varepsilon}$ div $\underline{\Phi}\, dx\, dy\, dz$. Dividing by the volume of B_ε, letting ε approach 0, and using the continuity of div $\underline{\Phi}$ at P_0 gives $\lim_{\varepsilon\to 0}$ [(flux of $\underline{\Phi}$ through ∂B_ε)/volume of B_ε] = (div $\underline{\Phi}$)(P_0). We note that the roundness of B_ε is not essential. The result is true for any regions which shrink down to the point P_0.

57. (a) Using Stokes' theorem, we let $P\,dx + Q\,dy + R\,dz$ be a differential form defined in space. Let C be the path composed of C_1 from $(0,0,0)$ to $(x,0,0)$, C_2 from $(x,0,0)$ to $(x,y,0)$ and C_3 from $(x,y,0)$ to (x,y,z). Then set $u = f(x,y,z) = \int_C (P\,dx + Q\,dy + R\,dz)$ and define $P = a(x,y,z)$, $Q = b(x,y,z)$, $R = c(x,y,z)$. We have $u = \int_0^x a(t,0,0)dt + \int_0^y b(x,t,0)dt + \int_0^z c(x,y,t)dt$, so $\partial u/\partial z = R$. Define another path through C_1 and then through C_4 from $(x,0,0)$ to $(x,0,z)$, followed by C_5 from $(x,0,z)$ to (x,y,z). Then define $u' = \int_0^x a(t,0,0)dt + \int_0^z c(x,0,t)dt + \int_0^y b(x,t,z)dt$, so $\partial u'/\partial y = Q$. We can apply Green's theorem to the rectangle D bounded by C_2, C_3, C_4, and C_5 to get $\iint_D (\partial Q/\partial y - \partial R/\partial z)dy\,dz = \int_{C_2+C_3+(-C_4)+(-C_5)} (Q\,dy + R\,dz) = \int_{C_1+C_2+C_3} (Q\,dy + R\,dz) - \int_{C_1+C_4+C_5} (Q\,dy + R\,dz) = u - u'$. $\underline{\nabla} \times \underline{\Phi} = \underline{0}$ implies $\partial R/\partial y = \partial Q/\partial z$, so the double integral over D is 0 and $u = u'$; therefore, $\partial u/\partial y = Q$.

Now, let C be composed of C_6 from $(0,0,0)$ to $(0,y,0)$, C_7 from $(0,y,0)$ to $(x,y,0)$, and C_8 from $(x,y,0)$ to (x,y,z). Define $u'' = \int_0^y b(0,t,0)dt + \int_0^x a(t,y,0)dt + \int_0^z c(x,y,t)dt$, so $\partial u''/\partial z = R$. Then, let C be composed of C_6, C_9 from $(0,y,0)$ to $(0,y,z)$, and C_{10} from $(0,y,z)$ to (x,y,z). Define $u''' = \int_0^y b(0,t,0)dt + \int_0^z c(0,y,t)dt + \int_0^x a(t,y,z)dt$, so $\partial u'''/\partial x = P$. Apply Green's theorem to the rectangle bounded by C_1, C_8, C_9, and C_{10}, and get $\iint_D (\partial P/\partial x - \partial R/\partial z)dx\,dz = \int_{C_7+C_8+(-C_9)+(-C_{10})} (P\,dx + R\,dz) = \int_{C_6+C_7+C_8} (P\,dx + R\,dz) - \int_{C_6+C_9+C_{10}} (P\,dx + R\,dz) = u'' - u'''$. $\underline{\nabla} \times \underline{\Phi} = \underline{0}$ implies $\partial P/\partial z = \partial R/\partial x$, so the double integral over D is 0 and $u'' = u'''$;

57. (a) (continued)

therefore $u''' = u$ becuase $\partial u''/\partial z = R = \partial u/\partial z$. Consequently, $\partial u/\partial x = P$ and $P\underline{i} + Q\underline{j} + R\underline{k} = \underline{\nabla f} = \underline{\Phi}$ if $\underline{\nabla} \times \underline{\Phi} = \underline{0}$.

(b)
$$\underline{\nabla} \times \underline{F} = \begin{vmatrix} \partial/\partial x & \partial/\partial y & \partial/\partial z \\ 2xyz + \sin x & x^2z & x^2y \\ \underline{i} & \underline{j} & \underline{k} \end{vmatrix} = (x^2 - x^2)\underline{i} - (2xy - 2xy)\underline{j} +$$

$(2xz - 2xz)\underline{k} = \underline{0}$; therefore, $\underline{F}$ is a gradient by part (a). Integrating $2xyz + \sin x$ with respect to x yields $x^2y^z - \cos x + C(y,z)$. Taking the partials of this expression yields x^2z and x^2y , the $\underline{j}$ and $\underline{k}$ components respectively. Thus $f(x,y,z) = x^2yz - \cos x + C$.

61. By the divergence theorem, $\iiint_W \operatorname{div}(\underline{\nabla} \times \underline{\Phi})dx\,dy\,dz = \iint_{\partial W}(\underline{\nabla} \times \underline{\Phi})\cdot\underline{n}\,dA$. By definition, $\iiint_W \operatorname{div}(\underline{\nabla} \times \underline{\Phi})dx\,dy\,dz = \iiint_W \underline{\nabla}\cdot(\underline{\nabla} \times \underline{\Phi})dx\,dy\,dz$, which is zero by part (c). W is the region inside the two surfaces and $\partial W = S$ is the surface boundary, which is the union of the two surfaces. So since $\iint_{\partial W}(\underline{\nabla} \times \underline{\Phi})\cdot\underline{n}\,dA = 0$, so does $\iint_S(\underline{\nabla} \times \underline{\Phi})\cdot\underline{n}\,dA$. Finally, by Stokes' theorem, the line integral along the given closed curve, $\int_C \underline{\Phi}(\underline{r})\cdot d\underline{r}$, is also zero.

TEST FOR CHAPTER 18

1. True or false.

   (a) Green's theorem cannot be applied to a region described in polar coordinates by $1 \leqslant r \leqslant 2$ and $0 \leqslant \theta \leqslant 2\pi$.

   (b) A vector field is a gradient if and only if it is conservative.

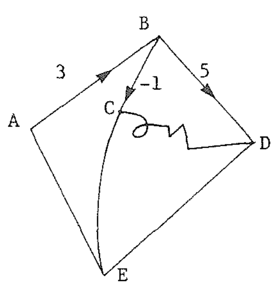
   (c) The curl of a vector field is another vector field.

1. (d) The normal vector in the formula for Gauss' divergence theorem must be a unit vector.

   (e) Suppose that a particle moves from point A to point B along two different paths C_1 and C_2. Then $\int_{C_1}\underline{\Phi}\cdot d\underline{r} = \int_{C_2}\underline{\Phi}\cdot d\underline{r}$.

2. Using standard notation as in the text, match the expression on the right with the expression on the left. Depending on the context, $\underline{V}$ may be either $P\underline{i} + Q\underline{j} + R\underline{k}$ or $P\underline{i} + Q\underline{j}$.

(a) $\iint_D(\partial P/\partial x + \partial Q/\partial y)\,dx\,dy$	(i) $\int_{\partial D}(P\,dx + Q\,dy)$
(b) $\iint_D(\partial Q/\partial x - \partial P/\partial y)\,dx\,dy$	(ii) $\int_{\partial D}\underline{V}(\underline{r})\cdot d\underline{r}$
(c) $\iint_D(\underline{\nabla} \times \underline{V})\cdot\underline{n}\,dA$	(iii) $\int_{\partial D}\underline{V}\cdot\underline{n}\,ds$

3. 

Let $\underline{\Phi}$ be a conservative vector field in the plane. The line integrals of $\underline{\Phi}$ along the curves from A to B, from B to C, and from B to D are 3, -1, and 5, respectively. If possible, find the line integrals of $\underline{\Phi}$ along the given paths:

   (a) From C to D

   (b) From D to E to A

   (c) From C to A

   (d) From E to C

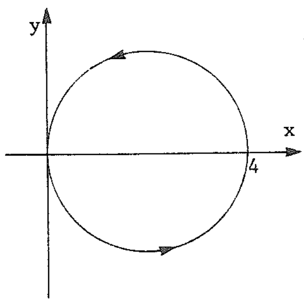
4. (a) Compute the surface integral of $\underline{W} = -x\underline{i} + y\underline{j} + 3xy\underline{k}$ over the surface $x^2 + y^2 = z$. Let the domain of z be the region $0 \leqslant x \leqslant 2$, $0 \leqslant y \leqslant x^2$.

   (b) Find div $\underline{Q}$ if $\underline{Q} = (x^2 + 3xy)\underline{i} - (2x + 3y + 3e^y)\underline{j}$.

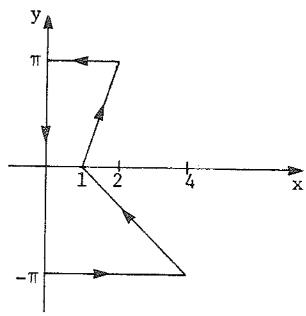
   (c) Find curl $\underline{P}$ if $\underline{P} = (x + y)\underline{i} + 2xy\underline{j} - 3(e^x - e^y)\underline{k}$.

5. Compute $\int_C (6y\,dx + 5x\,dy)$ for the following two curves, which are traversed once as shown:

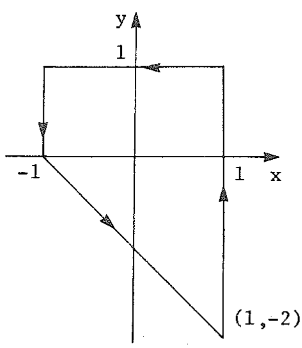
(a)



(b)



6. (a)



Compute the line integral of $\underline{\Phi} = [x^3/(x^4 + y^4)^2]\underline{i} + [y^3/(x^4 + y^4)^2]\underline{j}$ over the curve shown at the left.

(b) Show that $\partial[x^3/(x^4 + y^4)^2]/\partial y = \partial[y^3/(x^4 + y^4)^2]/\partial y$.

(c) Do your results verify or contradict any theorem? Explain.

7. Suppose that $\iint_S (\underline{\nabla} \times \underline{\Phi})\cdot\underline{n}\,dA = 6$, where S is the hemisphere $x^2 + y^2 + z^2 = 1$, $z \geqslant 0$.

(a) What is $\iint_S (\underline{\nabla} \times \underline{\Phi})\cdot\underline{n}\,dA$ if S is the sphere $x^2 + y^2 + z^2 = 1$?

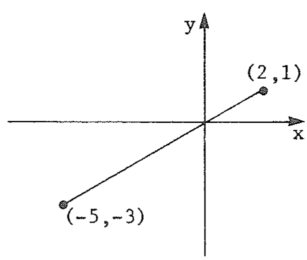
(b) What is $\iint_S (\underline{\nabla} \times \underline{\Phi})\cdot\underline{n}\,dA$ if S is the semi-ellipsoid $x^2 + y^2 + 4z^2 = 1$, $z \leqslant 0$?

8. Solve the exact differential equations or explain why is it not exact:

(a) $(-3x^2y\sin(x^3y) - 1/x) + (6 - x^3\sin(x^3y))dy/dx = 0$; $y(1) = \pi$.

(b) $(6 - x^3\sin(x^3y)) + (-3x^2y\sin(x^3y) - 1/x)dy/dx = 0$; $y(1) = \pi$.

9. (a) A region in space has volume $3e\pi$. Find the flux of $3z^2\underline{i} + (6y - 3x)\underline{j} + (z + 5xy^2)\underline{k}$ across the boundary of the surface, if possible. If not, explain why not.

(b) 

If $\underline{\Phi} = 3x\underline{i} + 2y^2\underline{j}$ and C is the curve shown at the left, find the line integral of $\underline{\Phi}$ along C , if possible. If not, explain why not.

10. Your butler comes to you seeking a raise, explaining that he needs to support his elderly mother, his wife, and his five kids. You refuse because you know that a force vector field equal to $\underline{\nabla f}$ exists throughout the house and is pushing him along. He claims that he works very hard.

(a) Approximately how much work does he do if $\underline{\nabla f} = 5x\underline{i} + 4y\underline{j}$ and he enters and leaves through the back door at $(-2,3)$?

(b) Explain your reasoning for your answer in part (a).

ANSWERS TO CHAPTER TEST

1. (a) False; the region may be divided as in Fig. 18.4.5.

(b) True

(c) True

(d) True

(e) False; only if $\underline{\Phi}$ is conservative.

2. (a) iii

   (b) i

   (c) ii

3. (a) 6

   (b) -8

   (c) -2

   (d) Unknown

4. (a) 3884/105

   (b) $2x + 3y - 3 - 3e^y$

   (c) $3e^y\underline{i} + 3e^x\underline{j} + (2y - 1)\underline{k}$

5. (a) -4π

   (b) -4π

6. (a) 3/32

   (b) Both equal $-8x^3y^3/(x^4 + y^4)^3$.

   (c) No; the cross-derivative test doesn't apply since $\underline{\Phi}$ is not defined at $(0,0)$.

7. (a) 0

   (b) -6

8. (a) $\cos(x^3y) + 6y - \ln|x| - 6\pi + 1 = 0$

   (b) Not exact; doesn't satisfy cross-derivative test.

9. (a) $21e\pi$

   (b) We don't know the sign of the line integral since the orientation isn't given.

10. (a) 0

    (b) Work is a line integral and $\int_C \underline{\nabla f}$ is independent of path. Starting and ending at the same point means that work is zero.

COMPREHENSIVE TEST FOR CHAPTERS 13 - 18 (Time limit: 3 hours)

1. True or false. If false, explain why.

   (a) On a given region in the plane, a minimum of a two-variable function f can occur only where $\partial f/\partial x = \partial f/\partial y = 0$.

   (b) The determinant of a 2×2 or 3×3 matrix A is positive whenever all of the entries of A are positive.

   (c) The Gaussian integral $\int_{-\infty}^{\infty} \exp(-4x^2)dx$ is equal to $\int_0^{\infty} \exp(-x^2)dx$.

   (d) A circle of radius 1, centered at the origin, has a larger curvature than a circle of radius 3, centered at $(7,0)$.

   (e) For any two vectors $\underline{p}$ and $\underline{r}$, we have $(\underline{p} \times \underline{r})\cdot\underline{p} = 0$.

   (f) A differentiable function of two variables, $f(x,y)$, has no critical points if either x or y is missing from the function.

   (g) The vector field $\underline{H}(x,y) = [e^{xy} + xye^{xy} - y^2\sin(xy) + 6x]\underline{i}$ + $[x^2e^{xy} + \cos(xy) - xy\sin(xy) - 3y^2]\underline{j}$ is a gradient.

   (h) Let $\underline{F}$ be a vector field and let C be a closed curve which is traversed once. Then $\int_C \underline{F}\cdot d\underline{r} = 0$.

   (i) The vector field $P\underline{i} + Q\underline{j}$ is a conservative field whenever $\partial P/\partial y = \partial Q/\partial x$.

   (j) The mixed partial $\partial^2 f/\partial x\partial y$ tells how fast the function $f(x,y)$ is changing along the line $x = y$.

2. Multiple choice.

   (a) If θ is the angle between $\underline{u}$ and $\underline{w}$, then $\|\underline{w} \times \underline{u}\|$ is:

      (i) $\|\underline{w}\|\cdot\|\underline{u}\|\cos\theta$.

      (ii) $\|\underline{w}\|\cdot\|\underline{u}\|\sin\theta$.

      (iii) $(\underline{w}\cdot\underline{u})/\|\underline{w}\|\cdot\|\underline{u}\|$.

      (iv) None of the above.

2. (b) If $f(x,y,z) = g(r,\theta,z) = h(\rho,\theta,\phi)$ and W, W', W^* represent the same region, then $\iiint_W f(x,y,z)\,dx\,dy\,dz$ equals:

(i) $\iiint_{W'} g(r,\theta,z)\,r\,dr\,d\theta\,dz$.

(ii) $\iiint_{W^*} h(\rho,\theta,\phi)\,\rho^2 \sin^2\phi\,d\rho\,d\theta\,d\phi$.

(iii) $\iiint_{W'} g(r,\theta,z)\,r\theta\,dr\,d\theta\,dz$.

(iv) $\iiint_{W^*} h(\rho,\theta,\phi)\,\rho\sin\theta\,d\rho\,d\theta\,d\phi$.

(c) The graph of $z - x^2 = 0$ in space is:

(i) A cylinder.

(ii) A hyperbolic paraboloid.

(iii) A hyperbola.

(iv) None of the above.

(d) If $y = f(x)$ and $z = F(x,y) = 0$, then dy/dx may be found implicitly by the formula:

(i) $(\partial z/\partial x)/(\partial z/\partial y)$.

(ii) $(\partial z/\partial y)/(\partial z/\partial x)$.

(iii) $-(\partial z/\partial x)/(\partial z/\partial y)$.

(iv) $-(\partial z/\partial y)/(\partial z/\partial x)$.

(e) The region described by $0 \leqslant x \leqslant 1$, $x^4 \leqslant y \leqslant \sqrt{x}$, $0 \leqslant z \leqslant 2 - x^2 - y^3$ is the same as the region described by:

(i) $0 \leqslant y \leqslant 1$, $y^4 \leqslant x \leqslant \sqrt{y}$, $0 \leqslant z \leqslant 2 - y^2 - x^3$.

(ii) $0 \leqslant z \leqslant 2$, $x^4 \leqslant y \leqslant \sqrt{x}$, $0 \leqslant x \leqslant 2 - z^2 - y^3$.

(iii) $0 \leqslant r \leqslant 1/\cos\theta$, $r^3\cos^4\theta \leqslant \sin\theta \leqslant r^{-1/2}\sqrt{\cos\theta}$, $0 \leqslant z \leqslant 2 - r^2\cos^2\theta - r^3\sin^3\theta$.

(iv) $0 \leqslant y \leqslant 1$, $y^2 \leqslant x \leqslant \sqrt[4]{y}$, $0 \leqslant z \leqslant 2 - x^2 - y^3$.

3. Short Answers.

(a) To apply Green's theorem on a region, what condition must hold on the region's boundary and what condition must hold for the integrand?

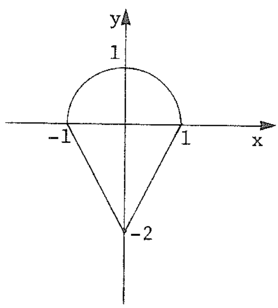
(b) Define a function of three variables.

(c) What conditions must be satisfied by partial derivatives to guarantee that $f(x_0,y_0)$ is a local maximum.

(d) Find the most general function $f(x,y)$ such that $\partial^2 f/\partial x\partial y = 3xy - y$.

(e) Define a local minimum of $f(x,y)$.

4.



The region at the left is bounded by a semicircle of radius 1 and two straight lines.

(a) Write the volume between this region and the graph of $f(x,y) = x^2 + y^2$ as a double integral.

(b) Compute the volume. [Hint: $\int x^2\sqrt{a^2 - x^2}\,dx = (x/8)(2x^2 - a^2)\sqrt{a^2 - x^2} + (a^4/8)\sin^{-1}(x/a)$, if $a > 0$.]

(c) Write the volume as a triple integral.

(d) Compute the surface integral of $\underline{\Phi} = (1/3)\underline{i} + y^2\underline{j} + 3y^4\underline{k}$ over the surface $f(x,y) = x^3 + y^3 - y$. Let the domain be the region sketched above.

5. (a) Sketch the oriented curve $C: (t, t, \sin t\pi)$, $-1 \leqslant t \leqslant 3/2$.

(b) What is the acceleration vector along C ?

(c) If $\underline{\Omega} = xy\underline{i} - yz\underline{j} + \underline{k}$, compute the line integral of $\underline{\Omega}$ along C .

(d) If $\underline{\Phi} = \underline{\nabla} f$, where $f(x,y,z) = xy^2 + 5e^z - 3 \cos z\pi + 4$, what is the line integral of $\underline{\Phi}$ along C ?

6. Let $f(x,y) = (x^2 + 3y^2)\exp(1 - x^2 - y^2)$.

(a) Classify all critical points of f .

(b) Does f have a global minimum or maximum? If yes, find them.

(c) Suppose that the domain of f is $[-2,2] \times [-1/2,1/2]$. Now, does f have a global minimum or maximum? If yes, find them.

7. (a) If $\underline{r} = x\underline{i} + y\underline{j} + z\underline{k}$ and f is a function of x, y, and z , how is the tangent plane defined by using $\underline{\nabla} f$?

(b) Use the definition from part (a) to find the tangent plane of $x^2 + y^2 + z^2 = 10$ at the point $(1,3,0)$.

(c) Suppose that g is a function of x, y, and z , and that x, y, and z are functions of t . Let $\underline{r} = x\underline{i} + y\underline{j} + z\underline{k}$. How is dg/dt defined in terms of the gradient?

(d) Use the definition from part (c) to compute dg/dt if $g(x,y,z) = xy + z^2$, $x = e^t$, $y = t + \sin t$, and $z = 1/t$.

8. (a) Write $\int_C (x^3y^2\, dx + x^3\cos(y^2)\, dy)$ as a double integral over D , where C is the boundary (oriented counterclockwise) of D .

(b) Compute $\iint_S (\underline{\nabla} \times \underline{B})\cdot\underline{n}\, dA$, where $\underline{B} = xy\underline{i} + y\underline{j} + z\underline{k}$ and S is the part of the unit sphere lying above the plane $z = 1/2$.

(c) Use Green's theorem to find the area of the region between $y = x^2 - 1$ and the parametrized curve, $x = \cos t$, $y = 2 \sin t$, $0 \leqslant t \leqslant \pi$.

9. (a) If $y = k(m,n)$, state the definition of $\partial y/\partial m$.

(b) Compute $\partial t/\partial p$ and $\partial t/\partial s$ if $t(m,p,r,s,u,v,z) = \exp(\cos rv) - \sin(\ln z) + mu/r - 1/p$.

(c) Let $f(x,y,z) = y - 2xz^2$. Compute the directional derivative of f in the direction of $\underline{i} - 2\underline{j} + 2\underline{k}$ at $(x,y,z) = (1,1,1)$.

(d) Find the direction in which the directional derivative of $f(x,y,z) = y - 2xz^2$ at $(x,y,z) = (1,1,1)$ is maximized.

10. Miscellaneous problems.

(a) Find the flux of $\underline{K} = (3x + y + z)\underline{i} + (x + 5y + z^2)\underline{j} + (x - y^3 + 6z)\underline{k}$ across the parallelepiped spanned by the vectors $\underline{i} + \underline{j} + \underline{k}$, $2\underline{i} - \underline{k}$, and $3\underline{i} + \underline{j} - 2\underline{k}$.

(b) What is curl $\underline{K}$ for $\underline{K}$ as in part (a)?

(c) Sketch the graph of $2^2 + (-1)^2 + z^2 = 25$.

(d) Write down $\partial(x,y,z)/\partial(\rho,\theta,\phi)$ for x, y, and z defined by spherical coordinates.

11. Extra Credit. Choose the best answer.

Calculus is ____________________ .

(a) a lot more enjoyable than going through a torture chamber.

(b) a fun hobby for A students.

(c) a combination of two four-letter words.

(d) worse than an evening with the in-laws.

ANSWERS TO COMPREHENSIVE TEST

1. (a) False; the minimum may occur on the boundary of the region.

   (b) False; let $A = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$.

   (c) True

   (d) True

   (e) True

   (f) True

   (g) True

   (h) False; $\underline{F}$ must be conservative to guarantee that $\int_C \underline{F}\cdot d\underline{r} = 0$.

   (i) False; $P\underline{i} + Q\underline{j}$ must be defined throughout the domain.

   (j) False; the directional derivative gives that information.

2. (a) ii

   (b) i

   (c) i

   (d) iii

   (e) iv

3. (a) The boundary must be traversed counterclockwise; the partials must be continuous on the region.

   (b) A rule which assigns a single value to each point (x,y,z) in the domain.

   (c) At (x_0,y_0) , $f_x = f_y = 0$; $f_{xx} < 0$; and $f_{xx}f_{yy} - f_{xy}^2 < 0$.

   (d) $3x^2y^2/4 - xy^2/2 + g(x) + h(y) + \text{constant}$, where g is a function of x only and h is a function of y only.

   (e) (x_0,y_0) is a local minimum point of f if there is a disk about (x_0,y_0) such that $f(x,y) \geqslant f(x_0,y_0)$ for all (x,y) in the disk.

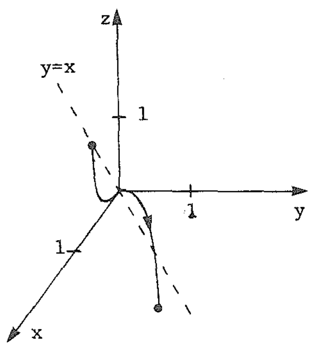
4. (a) By symmetry, $V = 2\int_0^1 \int_{2x-2}^{\sqrt{1-x^2}} (x^2 + y^2)\,dy\,dx$.

(b) $\pi/4 + 5/3$

(c) $V = \int_0^1 \int_{2x-2}^{\sqrt{1-x^2}} \int_0^{x^2+y^2} dz\,dy\,dx$

(d) $-\pi/4 - 5/3$

5. (a)



(b) $(0,0,-\pi^2 \sin \pi t)$, $-1 \leqslant t \leqslant 3/2$

(c) $35/24 + 1/\pi^2$

(d) $11/8 + 5e^{-1}$

6. (a) $(0,0)$ = local minimum; $(0,\pm 1)$ = local maximum; $(\pm 1,0)$ = saddle point

(b) Minimum: $(0,0,0)$; maximum: $(0,\pm 1,3)$

(c) Minimum: $(0,0,0)$; maximum: $(\pm 1/2,\pm 1/2,\sqrt{e})$

7. (a) $\underline{\nabla} f(\underline{r}_0)\cdot(\underline{r} - \underline{r}_0)$

(b) $x + 3y = 10$

(c) $\underline{\nabla} f(\underline{r})\cdot(d\underline{r}/dt)$

(d) $(1 + t + \sin t + \cos t)e^t - 2/t^3$

8. (a) $\iint_D (3x^2 \cos(y^2) - 2x^3 y)\,dx\,dy$

(b) 0

(c) $4/3 + \pi$

9. (a) $\lim_{\Delta m \to 0} \{[k(m + \Delta m, n) - k(m,n)]/\Delta m\}$

(b) $\partial t/\partial p = 1/p^2$; $\partial t/\partial s = 0$

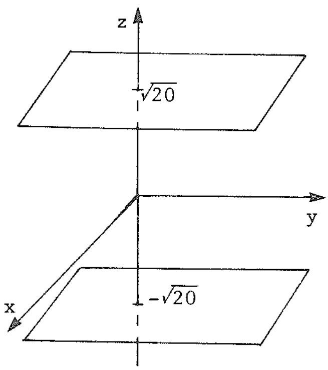
(c) -4

(d) $-2\underline{i} + \underline{j} - 4\underline{k}$

10. (a) 52

(b) $(-3y^2 - 2z)\underline{i}$

(c)



(d) $\begin{bmatrix} \sin\phi\cos\theta & -\rho\sin\phi\sin\theta & \rho\cos\phi\cos\theta \\ \sin\phi\sin\theta & \rho\sin\phi\cos\theta & \rho\cos\phi\sin\theta \\ \cos\phi & 0 & -\rho\sin\phi \end{bmatrix}$